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DEHN FILLINGS OF LARGE HYPERBOLIC 3-MANIFOLDS

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Abstract

Let M be a compact, connected, orientable, hyperbolic 3-manifold whose boundary is a torus and which contains an essential closed surface S. It is conjectured that 5 is an upper bound for the distance between two slopes on ∂M whose associated fillings are not hyperbolic manifolds. In this paper we verify the conjecture when the first Betti number of M is at least 2 by showing that given a pseudo-Anosov mapping class f of a surface and an essential simple closed curve γ in the surface, then 5 is an upper bound for the diameter of the set of integers n for which the composition of f with the n^{th} power of a Dehn twist along γ is not pseudo-Anosov. This sharpens an inequality of Albert Fathi. For large manifolds M of first Betti number 1 we obtain partial results. Set

$$\mathcal{C}(S) = \{\text{slopes } r \mid \ker(\pi_1(S) \to \pi_1(M(r))) \neq \{1\}\}.$$

A *singular slope* for S is a slope $r_0 \in \mathcal{C}(S)$ such that any other slope in $\mathcal{C}(S)$ is at most distance 1 from r_0. We prove that the distance between two exceptional filling slopes is at most 5 if either (i) there is a closed essential surface S in M with $\mathcal{C}(S)$ finite, or (ii) there are singular slopes $r_1 \neq r_2$ for closed essential surfaces S_1, S_2 in M.

1. Introduction

Consider a compact, connected, orientable, irreducible 3-manifold M whose boundary is a torus. We shall assume throughout that M is *hyperbolic*. This means that its interior admits a complete hyperbolic metric of finite volume. A *slope* on ∂M is a ∂M-isotopy class of unoriented essential simple closed curves. As usual, $\Delta(r_1, r_2)$ denotes the

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distance between two slopes r_1 and r_2 on ∂M, i.e., their minimal geometric intersection number. The *diameter* of a set $\mathcal{S}$ of slopes is the quantity

$$\Delta(\mathcal{S}) = \max\{\Delta(r_1, r_2) \mid r_1, r_2 \in \mathcal{S}\} \in \{0, 1, 2, 3, \dots, \infty\}.$$

The *Dehn filling* of M with slope r is the manifold $M(r)$ obtained by attaching a solid torus V to M by a homeomorphism $\partial V \to \partial M$ which sends a meridian curve of V to a simple closed curve in ∂M of slope r. Thurston's hyperbolic Dehn surgery theorem implies that all but finitely many fillings of M are hyperbolic manifolds [31], and there has been a great deal of interest in describing the possible configurations for the set of *exceptional slopes*

$$\mathcal{E}(M) = \{r \mid M(r) \text{ is not hyperbolic}\}.$$

The second author has examined the known manifolds for which $\mathcal{E}(M)$ is large and it is interesting to note that they are all fillings of the Whitehead link exterior [13]. Consideration of these examples led him to the following conjecture.

Conjecture 1.1 (Gordon)**.** *If M is a compact, connected, orientable, hyperbolic 3-manifold whose boundary is a torus then $\#\mathcal{E}(M) \leq 10$ and $\Delta(\mathcal{E}(M)) \leq 8$. Moreover if W is the Whitehead link exterior, T a component of ∂W, and $M \not\cong W(T; -1), W(T; 5), W(T; 5/2)$, or $W(T; -2)$, then $\Delta(\mathcal{E}(M)) \leq 5$ and $\#\mathcal{E}(M) \leq 8$.*

A manifold M as above is called *large* if it contains a closed essential surface. Otherwise it is called *small*. It turns out that $W(T; -1)$, $W(T; 5), W(T; 5/2), W(T; -2)$ are each *small* (see the Appendix) and so it is expected that $\Delta(\mathcal{E}(M)) \leq 5$ whenever M is large, for instance when the first Betti number of M, denoted $b_1(M)$ below, is at least 2. We shall prove:

Theorem 1.2. *Let M be a compact, connected, orientable, hyperbolic 3-manifold whose boundary is a torus.*

(1) *If $b_1(M) \geq 2$, then $\Delta(\mathcal{E}(M)) \leq 5$ and $\#\mathcal{E}(M) \leq 7$.*

(2) *If $b_1(M) \geq 3$, then $\Delta(\mathcal{E}(M)) \leq 4$ and $\#\mathcal{E}(M) \leq 6$.*

It has proven difficult to analyze $\mathcal{E}(M)$ by methods of a purely differential geometric nature and topologists have adopted an approach related to Thurston's hyperbolisation conjecture. This is what we do here. Consider the set of *topologically exceptional slopes*

$$\mathcal{E}_{TOP}(M) = \{r \mid M(r) \text{ is reducible, toroidal,} \\ \text{or Seifert fibred, or } \pi_1(M(r)) \text{ is finite}\}.$$

It is well-known that $\mathcal{E}_{TOP}(M) \subseteq \mathcal{E}(M)$ and the hyperbolisation conjecture asserts that these two sets are, in fact, equal. When $b_1(M) \geq 2$, Thurston's hyperbolisation theorem for Haken manifolds [32, Theorem 2.5] (see also Chapter VIII of [1]) implies that $M(r)$ is hyperbolic if and only if it contains no essential 2-spheres or tori, and is not Seifert fibred. The case when $M(r)$ is either reducible or toroidal can be understood through the application of known results. Our contribution to the proof of Theorem 1.2(1) deals with the possibility that $M(r)$ is Seifert fibred. A key special case arises when M is the exterior of a knot γ which lies in a fibre S of a locally trivial surface bundle over the circle with smooth monodromy $f : S \to S$. Let $T_\gamma : S \to S$ denote a Dehn twist along γ. In this setting, the exceptional surgery problem translates into understanding the set

$$\mathcal{N}(f, \gamma) = \{n \mid T_\gamma^n f \text{ is not a pseudo-Anosov mapping class}\}.$$

Fathi [8] has shown that $\mathcal{N}(f, \gamma)$ has diameter at most 6 by studying the action of the mapping class group of S on its space of measured laminations. In order to prove Part (1) of Theorem 1.2, it is necessary for us to improve his result by 1.

Theorem 1.3. *Let S be a closed connected orientable surface of positive genus. Suppose that $f : S \to S$ is a pseudo-Anosov diffeomorphism and γ is a simple closed essential curve in S. Then the set of integers n for which $T_\gamma^n f$ is not pseudo-Anosov has diameter at most* 5.

It seems reasonable to expect that the diameter of $\mathcal{N}(f, \gamma)$ is at most 4. For instance, it is easy to see that this is the case when S is a torus. Further, Fathi derived this bound in the case where γ is a separating curve ([8, Theorem 5.4]. See also Inequality 2.3). Our next result provides further evidence in the case where 1 is an eigenvalue of $f_* : H_1(S) \to H_1(S)$.

Theorem 1.4. *Let S be a closed connected orientable surface of positive genus. Suppose that $f : S \to S$ is a pseudo-Anosov diffeomorphism and γ is a simple closed essential curve in S. Let f_* be the automorphism of $H_1(S)$ induced by f and suppose that $|f_* - I| = 0$. Then* *the set of integers n for which $T_\gamma^n f$ is not pseudo-Anosov has diameter at most* 4*.*

In the final sections of the paper we consider the case where M is large, though allowing the possibility that $b_1(M) = 1$. Given a closed, essential surface S in M, set

$$\mathcal{C}(S) = \{r \mid S \text{ compresses in } M(r)\}.$$

A *singular slope* for S is a slope r_0 on ∂M such that S compresses in $M(r_0)$ but stays incompressible in $M(r)$ if $\Delta(r, r_0) > 1$. By Wu's theorem (Theorem 6.1), a singular slope for S exists as long as $\mathcal{C}(S) \neq \emptyset$. Moreover:

- A singular slope for S is unique if $\mathcal{C}(S)$ is infinite.
- Each slope in $\mathcal{C}(S)$ is a singular slope for S if $\mathcal{C}(S)$ is finite.

It turns out that the slopes in $\mathcal{E}(M)$ are located close to singular slopes for surfaces.

Theorem 1.5. *Let M be a compact, connected, orientable, hyperbolic* 3*-manifold whose boundary is a torus and suppose that r_0 is a singular slope of a closed essential surface $S \subset M$. Then*

$$\Delta(r_0, r) \leq \begin{cases} 1 & \text{if } M(r) \text{ is either small or reducible} \\ 1 & \text{if } M(r) \text{ is Seifert and } S \text{ does not separate} \\ 2 & \text{if } M(r) \text{ is toroidal and } \mathcal{C}(S) \text{ is infinite} \\ 3 & \text{if } M(r) \text{ is toroidal and } \mathcal{C}(S) \text{ is finite.} \end{cases}$$

Since Haken manifolds satisfy the hyperbolisation conjecture and closed Seifert manifolds are either small, or reducible, or toroidal, or contain non-separating horizontal surfaces, the following is immediate.

Corollary 1.6. *Suppose that r_0 is a singular slope of a closed essential surface $S \subset M$ and $r \in \mathcal{E}(M)$. Then*

$$\Delta(r_0, r) \leq \begin{cases} 2 & \text{if } \mathcal{C}(S) \text{ is infinite} \\ 3 & \text{if } \mathcal{C}(S) \text{ is finite.} \end{cases}$$

There are several topologically significant situations when the existence of a closed essential surface and associated singular slope r_0 are guaranteed by conditions on the filling $M(r_0)$. Here is one such. Manifolds which admit Seifert structures whose base orbifolds are either a

2-sphere with at most three cone singularities, or a projective plane with at most one cone singularity, are called *small Seifert* manifolds. Otherwise they are called *big Seifert*. They are called *very big Seifert* if they are big Seifert but do not have a base orbifold of the form $P^2(p,q)$, or $S^2(2,2,2,2)$, or the Klein bottle K. Evidently the generic Seifert fibred space is very big.

Theorem 1.7. *Let M be a compact, connected, orientable, hyperbolic 3-manifold whose boundary is a torus. Suppose that $M(r_0)$ is a big Seifert fibred manifold whose base orbifold $\mathcal{B}$ is not of the form $P^2(p,q)$. If $\mathcal{B}$ is the Klein bottle or $S^2(2,2,2,2)$, assume that $b_1(M) \geq 2$. Then r_0 is a singular slope of a closed essential surface $S \subset M$.*

The reader may notice that the role of a singular slope of a surface in Theorem 1.5 is reminiscent of that of degeneracy slopes of branched surfaces in theorems from the theory of laminations. Let B be an essential branched surface in M. We call a slope r_0 on ∂M the *degeneracy slope* of B if B is disjoint from ∂M and some component of the exterior $E(B)$ of B is a collar $T \times I$ on $\partial M = T \times \{0\}$ with a nonempty set of cusps on $T \times \{1\}$ whose slope corresponds to r_0 on $T \times \{0\}$. This condition on $E(B)$ implies that B remains essential in $M(r)$ whenever $\Delta(r_0, r) \geq 2$.

Theorem 1.8 ([38, Theorem 2.5])**.** *If r_0 is a degeneracy slope for some essential branched surface in M and $r \in \mathcal{E}_{TOP}(M)$, then $\Delta(r_0, r) \leq 2$.*

There are various conditions under which the existence of degeneracy slopes has been verified. For instance this will be the case when $b_1(M) > 1$ ([10]) or when M fibres over the circle with pseudo-Anosov monodromy ([12, Theorem 5.3]). Gabai and Mosher have claimed the existence of appropriate essential branched surfaces and degeneracy slopes in general, though we shall not use this.

Theorem 1.9. *Suppose that M is a compact, connected, orientable, hyperbolic 3-manifold with $b_1(M) = 1$.*

(1) *If there is a closed, essential surface $S \subset M$ such that $\mathcal{C}(S)$ is finite, then $\Delta(\mathcal{E}(M)) \leq 5$.*

(2) *If there are at least two different slopes on ∂M each of which is a singular slope of an essential closed surface, then $\Delta(\mathcal{E}(M)) \leq 5$.*

(3) *If there are at least two different slopes on ∂M each of which is ei-*

ther a singular slope of an essential closed surface or a degeneracy slope of an essential branched surface, then $\Delta(\mathcal{E}_{TOP}(M)) \leq 5$.

According to Theorem 1.7, a very big Seifert filling slope on ∂M is a singular slope of a closed essential surface.

Corollary 1.10. *Suppose that M is a compact, connected, orientable, hyperbolic 3-manifold with a torus boundary. If M has two very big Seifert fillings, then* $\Delta(\mathcal{E}(M)) \leq 5$.

There are various open conjectures concerning Seifert surgery on a hyperbolic knot K in the 3-sphere. For instance it is thought that a nontrivial Seifert surgery slope r on such a knot is integral; this means that $\Delta(r, \mu_K) = 1$ where μ_K is the meridional slope of K. It is known that the only Seifert manifolds which could possibly arise as nontrivial, non-integral surgery on a hyperbolic knot are small and have base orbifolds of the form $S^2(p, q, r)$ where $p, q, r \geq 2$ [2, Corollary 1.7]. It is also thought that no very big Seifert manifold can arise as surgery on a hyperbolic knot in S^3. We prove:

Theorem 1.11. *Suppose that K is a hyperbolic knot in the 3-sphere with exterior M_K. Suppose further that r is a non-meridional slope on ∂M_K such that $M_K(r)$ is Seifert fibred.*

(1) *If K is a small knot, then $M_K(r)$ is not a very big Seifert manifold.*

(2) *If r_0 is a singular slope of an essential closed surface in M_K, then* $\Delta(r_0, \mu_K) \leq 1$ *and* $\Delta(r_0, r) \leq 1$.

(3) *If μ_K is a singular slope of an essential closed surface in M_K, then r is an integral slope. In particular, this occurs if either μ_K is a boundary slope or there is an essential closed surface S in M_K such that $\mathcal{C}(S)$ is finite.*

(4) *If K admits a very big Seifert surgery slope r_0, then r_0 is integral and $\Delta(r_0, r) \leq 1$. Hence K admits no more than two very big Seifert surgeries, and if two, then:*

  - *They correspond to successive integral slopes.*
  - *If r is non-integral, it is half-integral.*

(5) *If K is amphicheiral and $M_K(r)$ is a big Seifert manifold, then K is fibred and r is the longitudinal slope.*

The paper is organized as follows. In §2 we analyze compositions of pseudo-Anosov diffeomorphisms with powers of a Dehn twist, consequently proving Theorem 1.3 and part of Theorem 1.4. To complete the proof of the latter theorem we must investigate the distance between toroidal filling slopes on the boundary of manifolds M with $b_1(M) \geq 3$. This is done in §3. In §4 we introduce the notions of hollow product and new annuli and prove, for instance, that often in the absence of the latter, we are working with the former. This will be of importance several times in the paper. Theorem 1.2 is dealt with in §5 and singular slopes associated to closed essential surfaces are examined in §6. In particular we prove Theorem 1.5 here. Sections 7 and 8 are devoted to the proofs of Theorem 1.9 and Theorem 1.11 respectively. In §9 we give some examples of non-hyperbolic Dehn fillings of large hyperbolic manifolds, illustrating how close our results are to being sharp. Finally we prove that most fillings of the Whitehead link exterior are small manifolds in the Appendix.

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2. Dehn twists, pseudo-Anosov diffeomorphisms and exceptional fillings

Let S be a closed, connected, orientable surface of positive genus and $f : S \to S$ an orientation preserving diffeomorphism. The mapping torus of f

$$W(f) = (S \times I)/\{(x, 1) = (f(x), 0)\}$$

is a locally-trivial S-bundle over the circle. It is straightforward to see that $W(f)$ is toroidal when f is reducible, and Seifert fibred when f is periodic. A major contribution of Thurston was to prove that $W(f)$ is a hyperbolic manifold if and only if the genus of S is larger than one and f is pseudo-Anosov [33] (see also [27]). In this section we investigate surgeries on a knot $K \subset W(f)$ which corresponds to an essential simple closed curve γ lying in a fixed fibre $S \subset W(f)$. Let M be the exterior of such a knot K in $W(f)$. A parallel of γ on S determines the *canonical slope* c of K. If we orient c and the meridian μ of K, their associated homology classes form an ordered basis $\{\alpha(\mu), \alpha(c)\}$ for $H_1(\partial M)$. Let $M(\frac{m}{n})$ denote the manifold obtained by filling M along the slope corresponding to $m\alpha(\mu) + n\alpha(c)$. The following lemma is due to Stallings [30].

Lemma 2.1 (Stallings)**.** *Let γ be an essential simple closed curve on S and K the associated knot in $W(f)$. Then $M(\frac{1}{n}) \cong W(T_\gamma^n f)$ where $T_\gamma : S \to S$ is a Dehn twist along γ.*

It follows then, in the case that genus$(S) > 1$, that $M(\frac{1}{n})$ is hyperbolic if and only if $T_\gamma^n f$ is pseudo-Anosov. Long and Morton [23] observed that when f is a pseudo-Anosov diffeomorphism, K is a hyperbolic knot (see Lemma 2.2 below), and hence Thurston's hyperbolic Dehn surgery theorem implies that the set of integers n for which $T_\gamma^n f$ is not pseudo-Anosov is finite.

Lemma 2.2 (Long-Morton)**.** *Let γ be an essential simple closed curve on S and let K be the associated knot in $W(f)$. If the genus of S is at least 2 and f is pseudo-Anosov, then K is a hyperbolic knot.*

Proof. Since $b_1(M) \geq 2$ it suffices to show that M is irreducible and atoroidal [32, Theorem 2.5]. Consider then an embedded 2-sphere $\Sigma \subset \mathrm{int}(M)$. Since $S \not\cong S^2$, any S-bundle over the circle is irreducible. In particular $\Sigma = \partial B$ where B is a 3-ball in $M(\frac{1}{0}) = W(f)$. Since γ is an essential curve in S, K cannot lie in the interior of B. Thus $B \subset M$ and so M is irreducible.

The fact that M is atoroidal is proved in Lemma 1.1 of [23]. They show that an essential torus in M may be isotoped so that its intersection with S yields an essential link in S invariant under f. This intersection can never be empty because there is no such torus in $S \times I$.

q.e.d.

Albert Fathi greatly sharpened the finiteness result of Long and Morton by showing that if f is pseudo-Anosov, though $T_\gamma^n f$ and $T_\gamma^m f$ are not, then $|n - m| \leq 6$ [8, Theorem 0.1]. Actually Fathi, following a suggestion of Francis Bonahon, observed that the condition that f be pseudo-Anosov can be relaxed to requiring that the pair (f, γ) *fills* S. This means that given any essential simple closed curve γ' on S, there is some $j \in \mathbb{Z}$ such that the geometric intersection number $i(\gamma', f^j(\gamma))$ is positive. Equivalently we may choose integers $j_1, j_2, \ldots, j_n$ and isotopic images of $f^{j_1}(\gamma), f^{j_2}(\gamma), \ldots, f^{j_n}(\gamma)$ which cut S into a family of 2-disks. For instance if f is irreducible, then given any essential γ, (f, γ) fills S.

Theorem 2.3 ([8, Theorem 5.1])**.** *Suppose that S is a closed, connected, orientable surface, γ an essential simple closed curve in S, and f an orientation preserving diffeomorphism of S such that (f, γ) fills S. If neither $T_\gamma^n f$ nor $T_\gamma^m f$ is pseudo-Anosov, then $|n - m| \leq 6$.*

Fathi's theorem is proved in the following fashion. Let $\mathcal{MF}(\mathcal{S})$ be the space of measured foliations on the surface S. He observes, after Thurston, that for a mapping class g of S to be pseudo-Anosov, it is necessary and sufficient for it to have no periodic points in $\mathcal{MF}(\mathcal{S})$, meaning that g has no finite orbits ([8, Theorem 2.1]). This property can be detected in an elementary fashion. It suffices to find a function $A : \mathcal{MF}(S) \to \mathbb{R}$ such that for any measured foliation $\mathcal{F}$, there are integers k, m and a constant $C \in (1, \infty)$ with $A(g^k(\mathcal{F})) \geq CA(\mathcal{F})$ and $A(g^m(\mathcal{F})) > 0$ ([8, Lemma 1.2]). Fathi's choice for this function is

$$A(\mathcal{F}) = i(\mathcal{F}, \gamma) \in [0, \infty)$$

where $i(\cdot, \cdot)$ denotes geometric intersection. He observes that since (f, γ) fills S, there is a least positive integer $k = k(f, \gamma)$ such that $i(\gamma, f^k(\gamma)) > 0$ and then proves that there is a real constant λ_0 such that for every $\mathcal{F} \in \mathcal{MF}(\mathcal{S})$ and every integer n, the inequality

$$A([T_\gamma^n f]^k(\mathcal{F})) + A([T_\gamma^n f]^{-k}(\mathcal{F})) \geq [|n - \lambda_0| - 1]i(\gamma, f^k(\gamma))A(\mathcal{F}) \tag{2.1}$$

holds [8, Proposition 5.2]. Consequently

$$\max\{A([T_\gamma^n f]^k(\mathcal{F})), A([T_\gamma^n f]^{-k}(\mathcal{F}))\} \geq \frac{[|n - \lambda_0| - 1]}{2} i(\gamma, f^k(\gamma))A(\mathcal{F}). \tag{2.2}$$

From this it is straightforward, using the criterion above, to show

Lemma 2.4 (cf. Proof of [8, Theorem 5.4])**.** *If*

$$\frac{[|n - \lambda_0| - 1]}{2} i(\gamma, f^k(\gamma)) > 1,$$

or equivalently $|n - \lambda_0| > 1 + \frac{2}{i(\gamma, f^k(\gamma))}$*, then* $T_\gamma^n f$ *is pseudo-Anosov.*

Hence if neither $T_\gamma^n f$ nor $T_\gamma^m f$ is pseudo-Anosov, then

$$|n - m| \leq 2 + \frac{4}{i(\gamma, f^k(\gamma))}. \tag{2.3}$$

This estimate immediately yields Theorem 2.3.

Next we present a mild refinement of Fathi's result which, when combined with work of the second author, allows us to improve Fathi's bound from 6 to 5 (see Theorem 2.6 below).

Lemma 2.5. *Using the notation developed above, if*

$$\frac{[|n - \lambda_0| - 1]}{2} i(\gamma, f^k(\gamma)) \geq 1,$$

then $T_\gamma^n f$ *is not a periodic mapping class.*

Proof. Set $g = T_\gamma^n f$ and observe that under our hypotheses, inequalities (2.1) and (2.2) become

$$A(g^k(\mathcal{F})) + A(g^{-k}(\mathcal{F})) \geq 2A(\mathcal{F})$$
$$\max\{A(g^k(\mathcal{F})), A(g^{-k}(\mathcal{F}))\} \geq A(\mathcal{F})$$

for every $\mathcal{F} \in \mathcal{MF}(\mathcal{S})$. Replacing $\mathcal{F}$ by $g^{mk}(\mathcal{F})$ we obtain

$$A(g^{(m+1)k}(\mathcal{F})) + A(g^{(m-1)k}(\mathcal{F})) \geq 2A(g^{mk}(\mathcal{F})) \tag{2.4}$$

and

$$\max\{A(g^{(m+1)k}(\mathcal{F})), A(g^{(m-1)k}(\mathcal{F}))\} \geq A(g^{mk}(\mathcal{F})) \tag{2.5}$$

for every $\mathcal{F} \in \mathcal{MF}(\mathcal{S})$ and every integer m. If g is periodic, there is an integer m_0 such that

$$A(g^{m_0 k}(\mathcal{F})) \geq A(g^{mk}(\mathcal{F})) \text{ for all } m. \tag{2.6}$$

The inequalities (2.4), (2.5) and (2.6) together imply the following equality

$$A(g^{m_0 k}(\mathcal{F})) = A(g^{(m_0+1)k}(\mathcal{F})) = A(g^{(m_0-1)k}(\mathcal{F})).$$

Inductively, this equality holds if m_0 is replaced by any integer m. In particular taking $m = 0$ gives

$$i(\gamma, \mathcal{F}) = i(\gamma, g^k(\mathcal{F})) = i(g^{-k}(\gamma), \mathcal{F}) \text{ for every } \mathcal{F} \in \mathcal{MF}(\mathcal{S}).$$

Since $i(\gamma, f^j(\gamma)) = 0$ for $1 \leq j < k$, we have $g^{-k}(\gamma) = [f^{-1}T_\gamma^{-n}]^k(\gamma) = f^{-k}(\gamma)$ and therefore

$$i(\gamma, \mathcal{F}) = i(f^{-k}(\gamma), \mathcal{F}) \text{ for every } \mathcal{F} \in \mathcal{MF}(\mathcal{S}).$$

It follows that $\gamma = f^k(\gamma)$ and thus $i(\gamma, f^k(\gamma)) = 0$, contrary to the definition of k. Thus g cannot be periodic. q.e.d.

Theorem 1.3 is a special case of our next result.

Theorem 2.6. *Let S be a closed connected orientable surface of positive genus. Suppose that $f : S \to S$ is a diffeomorphism and γ is a simple closed essential curve in S such that (f, γ) fills S. Then the set of integers n for which $T_\gamma^n f$ is not pseudo-Anosov has diameter at most 5.*

Proof. First suppose that S is a torus. The homomorphism which associates $g_* \in SL(H_1(S))$ to a mapping class g is an isomorphism. Further g is pseudo-Anosov if and only if $|\mathrm{tr}(g_*)| > 2$. It is straightforward to prove that since (f, γ) fills S, there is an integer $c \neq 0$ such that $\mathrm{tr}((T_\gamma^n f)_*) = \mathrm{tr}(f_*) + nc$, which implies the desired conclusion.

Next suppose that the genus$(S) > 1$ and that $T_\gamma^n f$ and $T_\gamma^m f$ are not pseudo-Anosov, where $|n-m| = 6$ (cf. Theorem 2.3). By Lemma 2.4,

$$\max\{|n - \lambda_0|, |m - \lambda_0|\} \leq 3$$

and so

$$6 = |n-m| \leq |n-\lambda_0| + |m - \lambda_0| \leq 6.$$

It follows that $|n - \lambda_0| = |m - \lambda_0| = 3$ and so by Lemma 2.5, neither $T_\gamma^n f$ nor $T_\gamma^m f$ is a periodic mapping class of S. They are reducible then and thus both $W(T_\gamma^n f)$ and $W(T_\gamma^m f)$ are toroidal manifolds. Let M be the exterior of the knot in $W(f)$ corresponding to γ, so that $W(T_\gamma^n f) = M(\frac{1}{n})$ and $W(T_\gamma^m f) = M(\frac{1}{m})$ (cf. Lemma 2.1). Since M is also homeomorphic to the exterior of the knot in $W(T_\gamma^j f)$ corresponding to γ, by choosing $j \gg n, m$ we may apply Lemmas 2.2 and 2.4 to see that M is hyperbolic. The second author proved [14, Theorem 1.2] that if the distance between two toroidal slopes on the boundary of a hyperbolic manifold is larger than 5, then that manifold has first Betti number 1. But this contradicts the fact that $b_1(M) \geq 2$ and $6 = |n-m| = \Delta(\frac{1}{n}, \frac{1}{m})$. Hence $|n - m| \leq 5$. q.e.d.

Theorem 1.4 follows from the following result.

Proposition 2.7. *Let S be a closed connected orientable surface of positive genus. Suppose that $f : S \to S$ is a diffeomorphism and γ is a simple closed essential curve in S such that (f, γ) fills S. Let f_* be the automorphism of $H_1(S)$ induced by f and suppose that $|f_* - I| = 0$. Then the set of integers n for which $T_\gamma^n f$ is not pseudo-Anosov has diameter at most* 4.

Proof. We shall assume that the genus of S is at least two, since the case when S is a torus was dealt with in the first paragraph of the previous proof. It was also noted in that proof that the exterior M of γ in $W(f)$ is a hyperbolic manifold and that there is a choice of basis for $H_1(\partial M) \cong \mathbb{Z}^2$ such that $M(\frac{1}{j}) \cong W(T_\gamma^j f)$.

Let $N(S) \subset W(f)$ be a collar neighborhood of S and set $W_0 = W(f) \setminus \mathrm{int}(N(S))$. Evidently $W_0 \cong S \times I$. The isomorphisms $H_j(W(f), S)$

$\cong H_j(W(f), N(S))$ (homotopy) $\cong H_j(W_0, \partial W_0)$ (excision) $\cong H_{j-1}(S)$ (Thom isomorphism) can be used to convert the exact sequence

$$H_2(W(f), S) \to H_1(S) \to H_1(W(f)) \to H_1(W(f), S) \\ \to H_0(S) \to H_0(W(f))$$

into an exact sequence

$$H_1(S) \xrightarrow{f_*-1} H_1(S) \to H_1(W(f)) \to \mathbb{Z} \to 0.$$

Thus $|f_* - I| = 0$ if and only if $b_1(W(f)) \geq 2$. Any Seifert fibred space whose first Betti number is larger than 1 admits an essential torus, and therefore if $T_\gamma^n f$ and $T_\gamma^m f$ are not pseudo-Anosov, both $W(T_\gamma^n f)$ and $W(T_\gamma^m f)$ are toroidal. The proposition is thus a consequence of Theorem 3.1, the main result of the next section. q.e.d.

3. Toroidal slopes on manifolds with large Betti number

The second author proved that the distance between toroidal filling slopes on the boundary of a large hyperbolic 3-manifold M is at most 5 [14]. In order to prove Proposition 2.7, it is necessary to improve this result by 1 under the assumption that $b_1(M) \geq 3$.

Theorem 3.1. *Let M be a compact, connected, orientable hyperbolic 3-manifold whose boundary is a torus and whose first Betti number is at least 3. Suppose that $M(r_1)$ and $M(r_2)$ are toroidal. Then $\Delta(r_1, r_2) \leq 4$.*

We will assume that M is as in the hypotheses of Theorem 3.1, and that $\Delta = 5$, and will show that this leads to a contradiction. Throughout we let $|X|$ denote the number of path components of a space X.

We use α or β to denote either 1 or 2, and when they appear together, then $\{\alpha, \beta\} = \{1, 2\}$.

Let V_α denote the filling solid torus in $M(r_\alpha)$. Amongst all essential tori in $M(r_\alpha)$, let T_α be one such that $|T_\alpha \cap V_\alpha|$ is minimal. Then $F_\alpha = T_\alpha \cap M$ is an essential punctured torus in M with boundary slope r_α. By an isotopy we may assume that no arc component of $F_1 \cap F_2$ is boundary parallel in F_α, and no circle component of $F_1 \cap F_2$ bounds a disk in F_α. As usual we define the intersection graph Γ_α in T_α, taking $T_\alpha \cap V_\alpha$ as vertices and arc components of $F_1 \cap F_2$ as edges. We assume that the reader is familiar with the basic terms and facts in this setting, such as the labeling and signs of vertices, the parity rule, Scharlemann cycles and their labelings, Scharlemann disks, S-cycles, extended S-cycles, positive (negative) edges, parallel edges, the labeling of endpoints of edges, the labeling of the corners of a face of Γ_α, the reduced graph $\hat{\Gamma}$ of a graph Γ, the edge class of an edge in Γ_α and its width. We take [14], [18], [37], and [3] as references. Let n_α be the number of vertices of Γ_α, or equivalently, the number of boundary components of F_α.

Lemma 3.2 ([3, Lemma 2.2 (1)]). *If Γ_α contains a Scharlemann cycle then T_β is separating, and hence n_β is even.*

Lemma 3.3. *Suppose that T_β separates $M(r_\beta)$, into X_1 and X_2. If Γ_α contains a Scharlemann cycle such that the corresponding Scharlemann disk lies in X_i, $i = 1$ or 2, then X_i is a $\mathbb{Q}$-homology $S^1 \times D^2$.*

Proof. Suppose without loss of generality that Γ_α contains a 12-Scharlemann cycle, and that the corresponding Scharlemann disk f lies in X_1. Let H_{12} be the 1-handle consisting of that part of V_β between fat vertices 1 and 2 of Γ_β on T_β, lying in X_1. Let $W = N(T_\beta \cup H_{12} \cup f)$. Then $\partial W = T_\beta \cup T'_\beta$, say, where T'_β is a torus, and W is a $\mathbb{Q}$-homology $T^2 \times I$. Moreover, $|T'_\beta \cap K_\beta| = n_\beta - 2$, and so, by the minimality of n_β, T'_β bounds a solid torus V' in $M(r_\beta)$. Then $X_1 = W \cup V'$ is a $\mathbb{Q}$-homology $S^1 \times D^2$. q.e.d.

Corollary 3.4. *Γ_α cannot have Scharlemann cycles lying on opposite sides of T_β.*

Proof. If it did, we would have $M(r_\beta) = X_1 \cup_{T_\beta} X_2$, where X_i is a $\mathbb{Q}$-homology $S^1 \times D^2$, $i = 1$ and 2. Hence $b_1(M(r_\beta)) \leq 1$, implying that $b_1(M) \leq 2$, contradicting our hypothesis on M. q.e.d.

Lemma 3.5.

(1) *If Γ_α has more than $n_\beta/2$ mutually parallel positive edges, then Γ_α has an S-cycle.*

(2) *If n_β is odd then Γ_α cannot have more than $(n_\beta - 1)/2$ mutually parallel positive edges.*

(3) *If $n_\beta \geq 4$ then Γ_α does not have an extended S-cycle.*

(4) *Γ_α cannot have more than $n_\beta/2+2$ mutually parallel positive edges.*

(5) *If Γ_α has $n_\beta/2 + 2$ mutually parallel positive edges then $n_\beta \equiv 0 \ (mod\ 4\ )$.*

(6) Γ_α *cannot have three S-disks with distinct label pairs lying on the same side of T_β.*

Proof. (1) This is [6, Corollary 2.6.7].

(2) This follows from (1) and Lemma 3.2.

(3) This is [3, Lemma 2.10] or [18, Theorem 3.2].

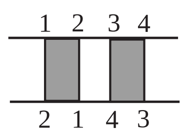
(4) If $n_\beta \neq 2$, this is [3, Lemma 2.11]. For the case $n_\beta = 2$, see (5) below.

(5) Suppose that Γ_α has a family of $n_\beta/2 + 2$ mutually parallel positive edges. Then n_β is even by (2). If $n_\beta \equiv 2 \pmod 4$ then, using (3) when $n_\beta > 2$, we see that the extremal bigons of the family are S-cycles lying on opposite sides of T_β. But this contradicts Corollary 3.4. (Cf. [37, Corollary 1.8].)

(6) This is [18, Theorem 3.5]. q.e.d.

Lemma 3.6. *Suppose $n_\beta = 4$, and that Γ_α has a disk face of odd order. Then Γ_α does not have four mutually parallel positive edges.*

Proof. Suppose that Γ_α has four mutually parallel positive edges. By Lemma 3.5 (3) we may assume that these edges are as shown



By Lemma 3.2, T_β is separating, and hence every face of Γ_α either has corners in $\{12, 34\}$ or in $\{23, 41\}$. Let g be a disk face of Γ_α of odd order.

Suppose that g is a $(12, 34)$-face. Then, without loss of generality, g has an odd number of 12-corners, and an even number (possibly zero) of 34-corners. But this contradicts [20, Lemma 5.13] (taking f and f_1 there to be the 12- and 34-S-cycles shown above, respectively, and $f_2 = g$).

Suppose that g is a $(23, 41)$-face. Then g and the $(23, 41)$-bigon face shown above represent linearly independent elements of the $\mathbb{Z}/2$-vector space on generators 23 and 41. Hence, if these faces lie in X_2, where $M(r_\beta) = X_1 \cup_{T_\beta} X_2$ and the 12-S-cycle lies in X_1, then it follows, as in the proof of Lemma 3.3, that X_2 is a $\mathbb{Z}/2$-homology $S^1 \times D^2$. This, together with Lemma 3.3, implies that $b_1(M(r_\beta)) \leq 1$, and hence that $b_1(M) \leq 2$, contrary to hypothesis. q.e.d.

Lemma 3.7. *Let Γ be a reduced graph on a torus with no vertex of valency less than 5. Then Γ has a 3-gon face.*

Proof. Let V, E and F be the number of vertices, edges and disk faces of Γ, respectively. Then $2E \geq 5V$. Assume that Γ has no 3-gon face. Then $2E \geq 4F$. Therefore

$$0 \leq V - E + F \leq \frac{2E}{5} - E + \frac{E}{2} = -\frac{E}{10},$$

a contradiction. q.e.d.

Lemma 3.8. *The vertices of Γ_α are not all of the same sign.*

Proof. Suppose that the vertices of Γ_α are all of the same sign. Write $n = n_\beta$.

First note that the reduced graph $\hat{\Gamma}_\alpha$ has no vertex of valency less than 5. For, if it did, we would have

$$5n \leq 4\left(\frac{n}{2} + 2\right)$$

by Lemma 3.5 (4), giving $n \leq 2$. But $n = 1$ is impossible (by the parity rule), while in the case $n = 2$ every disk face of Γ_α is a Scharlemann cycle, and we are done by Corollary 3.4.

Hence Γ_α has a 3-gon face, by Lemma 3.7. By a standard Euler characteristic argument, $\hat{\Gamma}_\alpha$ has a vertex of valency less than or equal to 6. Hence, by Lemma 3.5 (4),

$$5n \leq 6\left(\frac{n}{2} + 2\right),$$

and therefore $n \leq 6$. But $n = 6$ is impossible by Lemma 3.5 (5), $n = 1, 3$ or 5 is impossible by Lemma 3.5 (2), $n = 2$ is impossible, as argued above, by Corollary 3.4, and $n = 4$ is impossible by Lemma 3.6. q.e.d.

Lemma 3.9. *Γ_α has at most n_β mutually parallel negative edges.*

Proof. Suppose Γ_α has $n_\beta+1$ mutually parallel negative edges. Then by [14, Lemma 4.2] the corresponding permutation has only one orbit (note that $(M, \partial M)$ is not cabled since M is hyperbolic), and hence all the vertices of Γ_β have the same sign, contradicting Lemma 3.8. q.e.d.

It follows from Lemma 3.8 that $n_\alpha > 1$, $\alpha = 1, 2$. We will now proceed to show that $n_\alpha > 2$, $\alpha = 1, 2$.

Lemma 3.10. *Suppose $n_\beta = 2$. Then Γ_α cannot have more than two mutually parallel edges.*

Proof. Γ_α cannot have more than two mutually parallel positive edges by Lemma 3.5 (5).

Note that the vertices of Γ_β are of opposite sign, by Lemma 3.8. Hence if Γ_α has three mutually parallel negative edges, then the corresponding edges of Γ_β are loops, two at vertex 1 (say) and one at vertex 2. The two loops at vertex 1 are parallel in Γ_β. Hence we have edges that are parallel on both graphs, which contradicts [14, Lemma 2.1].

q.e.d.

Lemma 3.11. *Suppose $n_\beta = 2$, and that T_β separates $M(r_\beta)$, into X_1 and X_2. If Γ_α has a 3-gon face that lies in X_i, $i = 1$ or 2, then X_i is a $\mathbb{Z}/2$-homology $S^1 \times D^2$.*

Proof. Let f be a 3-gon face of Γ_α, and suppose without loss of generality that f lies in X_1. Let $H_{12} = V_\beta \cap X_1$, and let $W = N(T_\beta \cup H_{12} \cup f)$. Then $\partial W = T_\beta \cup T'_\beta$, where T'_β is a torus, and W is a $\mathbb{Z}/2$-homology $T^2 \times I$. Since $T'_\beta \cap K_\beta = \emptyset$, T'_β bounds a solid torus V' in $M(r_\beta)$. Hence $X_1 = W \cup V'$ is a $\mathbb{Z}/2$-homology $S^1 \times D^2$. q.e.d.

Proposition 3.12. $n_\beta \neq 2$.

Proof. For convenience write $m = n_\alpha$, $n = n_\beta$, and suppose $n = 2$.

Let E denote the number of edges of Γ_α, and F_k the number of disk faces of Γ_α of order k, $k \geq 2$. Each vertex of Γ_α has valency $\Delta n = 10$, so $E = 5m$.

Then

$$m - E + \sum_k F_k \geq \chi(T_\alpha) = 0,$$

giving

$$4m \leq \sum_k F_k. \tag{3.1}$$

We also have

$$2E \geq \sum_k kF_k,$$

giving

$$10m \geq \sum_k kF_k. \tag{3.2}$$

Multiplying (3.1) by 3 and subtracting (3.2) gives

$$2m \leq F_2 - \sum_{k \geq 4} (k-3)F_k.$$

Hence $F_2 \geq 2m$, and if $F_2 = 2m$ then $F_k = 0$ for $k \geq 4$.

We first show that one of the 2-gon faces of Γ_α is an S-cycle. So suppose not. Since $F_2 \geq 2m$, there exists a vertex v of Γ_α with at least four 2-gon faces of Γ_α incident to v. By Lemma 3.10, no two of these can share an edge, and hence we get four 1-edges and four 2-edges at v, which correspond to loops in Γ_β. Hence there is only one parallelism class of loops in Γ_β at vertex 1, containing four v-edges. It follows that Γ_β has $n_\alpha + 1$ parallel loops, contradicting either Lemma 3.5 (4) (when $n_\alpha > 2$) or Lemma 3.10 (when $n_\alpha = 2$).

By Lemma 3.2, T_β separates $M(r_\beta)$ into X_1 and X_2, and Γ_α has an S-cycle lying in X_1, say. By Corollary 3.4, Γ_α has no S-cycle in X_2.

Let G be the graph on T_α with the same vertices as Γ_α, and whose edges are in one–one correspondence with the F_2 2-gons of Γ_α, in the obvious way.

First suppose that $F_2 > 2m$. Then a simple Euler characteristic argument shows that G has a 2-gon or 3-gon face. A 2-gon face of G would give rise to three mutually parallel edges of Γ_α, contradicting Lemma 3.10. Let g be a 3-gon face of G. At least two of the vertices of g are of the same sign, and so the edge of g joining these two vertices corresponds to an S-cycle in Γ_α, lying in X_1. Hence g corresponds to a 3-gon face of Γ_α lying in X_2. Therefore X_2 is a $\mathbb{Z}/2$-homology $S^1 \times D^2$ by Lemma 3.11, and we get a contradiction as in the proof of Corollary 3.4.

Finally, suppose that $F_2 = 2m$. Then the only faces of Γ_α are 2-gons and 3-gons. Again, a simple Euler characteristic argument shows that all faces of G are 4-gons. Such a face corresponds to the union along an edge of two 3-gons faces of Γ_α. Thus again there is a 3-gon face of Γ_α in X_2, and we are done as before. q.e.d.

So from now on we shall assume that $n_\alpha > 2$, $\alpha = 1, 2$.

Lemma 3.13. *Suppose $n_\beta = 4$. Then Γ_α does not have four mutually parallel positive edges.*

Proof. By Lemma 3.9 and Lemma 3.5 (4), Γ_α cannot have more than n_β mutually parallel edges. Therefore $\hat{\Gamma}_\alpha$ has no vertex of valency less than 5, and hence Γ_α has a 3-gon face by Lemma 3.7. The conclusion now follows from Lemma 3.6. q.e.d.

Proposition 3.14. *$\hat{\Gamma}_\alpha$ has no vertex of valency less than or equal to* 5.

Proof. As noted above, no vertex of $\hat{\Gamma}_\alpha$ can have valency less than 5. Let u be a vertex of $\hat{\Gamma}_\alpha$ of valency 5. Then, by Lemma 3.5 (2) and (4), Lemma 3.13 and Lemma 3.9, each edge class at u is negative. Therefore all u-edges of Γ_β are positive. Let v be a vertex of $\hat{\Gamma}_\beta$ of valency at most 6. Since no two u-edges at v are parallel in Γ_β, there are at least 5 positive edge classes at v. Hence, writing $n = n_\alpha$, we have

$$5n \leq 5(n/2 + 2) + n,$$

giving $n \leq 6$. But $n = 6$ is impossible by Lemma 3.5 (5), $n = 3$ or 5 is impossible by Lemma 3.5 (2), and $n = 4$ is impossible by Lemma 3.13. q.e.d.

Proof of Theorem 3.1. By Lemma 3.8 and Proposition 3.12, we may assume that $n_\alpha > 2$, $\alpha = 1, 2$. By Proposition 3.14 we may assume that each vertex of $\hat{\Gamma}_\alpha$ has valency 6, $\alpha = 1, 2$. Since positive edges of Γ_α correspond to negative edges of Γ_β, we may assume that Γ_1 has at least as many positive edges as negative edges. Hence, writing $n = n_2$, there exists a vertex u of Γ_1 with at least $5n/2$ positive edges incident to it.

Using Lemma 3.5 (4) and (2), we see that at least three of the six edge classes of Γ_1 at u must be positive.

If there are at least five positive edge classes at u, then we get a contradiction, using Lemma 3.5 (2), (4) and (5), Lemma 3.9 and Lemma 3.13.

Suppose there are four positive and two negative edge classes at u. Then, by Lemma 3.5 (4) and Lemma 3.9, we get

$$5n \leq 4(n/2 + 2) + 2n,$$

giving $n \leq 8$. But $n = 3, 5$ or 7 are impossible by Lemma 3.5 (2), and $n = 6$ is impossible by Lemma 3.5 (5).

If $n = 4$, then by Lemma 3.13 and Lemma 3.9 the four positive classes have width 3, and the two negative classes have width 4. Then, without loss of generality, using Corollary 3.4, the positive classes together contain a 12-S-cycle, a 34-S-cycle, and a $(23, 41)$-bigon. Since Γ_1 also has a 3-gon face, the proof of Lemma 3.6 now gives a contradiction.

If $n = 8$, then the four positive edge classes must have width 6, and the two negative edge classes width 8. But then it is easy to see that the positive edge classes contain either an extended S-cycle, contradicting Lemma 3.5 (3), or three S-cycles on distinct label pairs lying on the same side of T_2, contradicting Lemma 3.5 (6).

Finally, consider the case where there are three positive and three negative edge classes at u. Then $5n/2 \leq 3(n/2 + 2)$, giving $n \leq 6$.

As before, $n = 6$ is impossible by Lemma 3.5 (5), and $n = 3$ or 5 by Lemma 3.5 (2). If $n = 4$, then there must be a positive edge class of width 4, contradicting Lemma 3.13. This completes the proof of Theorem 3.1. q.e.d.

Remark 3.15. The above proof of Theorem 3.1, with obvious modifications, also gives the following result: If M is a connected compact orientable hyperbolic 3-manifold whose boundary consists of $k > 3$ tori, then for any fixed boundary torus T of M, any two toroidal filling slopes of M along T have distance at most 4. Combining this with known results, it follows that 4 is also an upper bound for the distance between two non-hyperbolic Dehn filling slopes for M along T; see [15] for details.

4. Hollow products and new annuli

Throughout this section, S denotes a connected closed orientable surface of genus larger than one, $U = S \times [0, 1]$ is the product I-bundle over S, and $P : U \to S$ is the natural projection map. Note that every essential annulus in U is vertical, that is, isotopic to $P^{-1}(C)$ for some essential simple closed curve C in S [34].

If K is a knot in $\operatorname{int}(U)$ which is isotopic to the center circle of some essential annulus A_* in U, then its exterior $W = U - \operatorname{int}(N(K))$ is called a *hollow product*. Let A be one of the two components of $A_* \cap W$. Then A is an essential annulus in W with one boundary component in $\partial N(K)$ and the other on $\partial W \setminus \partial N(K)$. The slope c of the curve $A \cap \partial N(K)$ is called the *canonical slope* of W. If μ denotes the meridional slope of the knot, then $\Delta(c, \mu) = 1$. Obviously $W(c)$ is ∂-reducible and twisting along the annulus A shows that $W(r) \cong W(r')$ if $\Delta(r, c) = \Delta(r', c) = 1$. A simple homological calculation implies that if $\Delta(r, c) \neq \Delta(r', c)$, then $W(r) \not\cong W(r')$. In particular $W(r)$ is a product I-bundle if and only if $\Delta(r, c) = 1$.

Lemma 4.1. *Suppose that W is a hollow product with canonical slope c. Then:*

(1) *$W(r)$ is a product I-bundle if and only if $\Delta(r, c) = 1$.*

(2) *$W(r)$ is ∂-reducible if and only if $r = c$.*

Proof. Part (1) was observed above. Part (2) follows from [6, Theorem 2.4.3] (or Theorem 6.1 in the present paper) and Part (1). q.e.d.

Let W be a compact, connected, irreducible, orientable 3-manifold and r a slope on a toral boundary component of W. A *new annulus* in $W(r)$ is an essential annulus A such that W contains no annulus A' which has the same boundary as A. We are interested in situations where new annuli arise. Our next lemma leads to such situations.

Lemma 4.2. *Let $P : U = S \times I \to S$ be a product I-bundle. Let $F_0 \subset S$ be either a 2-disk or an essential annulus, and suppose that K is a knot in int(U) which can be isotoped off each essential annulus in $P^{-1}(S \setminus F_0)$. Then K is isotopic to a knot contained in $P^{-1}(F_0)$.*

Proof. Consider first the case where F_0 is a 2-disk. It is well known (see, e.g., [9, Lemme, p. 249]) that there are a transverse pair of non-isotopic essential simple closed curves C_1, C_2 in $S \setminus F_0$ which intersect minimally and such that each component of $S \setminus (C_1 \cup C_2)$ is an open disk. Denote by A_1, A_2 the essential annuli $P^{-1}(C_1), P^{-1}(C_2) \subset P^{-1}(S \setminus \text{int}(F_0))$. Our hypotheses allow us to suppose that $K \subset U \setminus A_1$ and to produce an isotopy of U, rel ∂, which moves A_2 to an annulus A_2' disjoint from K and transverse to A_1. No circle component of $A_1 \cap A_2'$ can be essential in A_1 or A_2' as C_1 and C_2 are not isotopic. Thus any circle component C of this intersection is inessential in both A_1 and A_2'. We may assume that C was chosen to be innermost in A_1 amongst all such circles. Hence if $D \subset A_1$ and $D' \subset A_2'$ are the disks bounded by C, then $D \cap D' = C$ so that $D \cup D'$ is a 2-sphere in U. The irreducibility of U implies that $D \cup D'$ is the boundary of a 3-ball $B \subset U$. Observe that we can isotope A_2' through B, rel the complement of an arbitrarily small neighborhood of D', so as to eliminate C from $A_1 \cap A_2'$. Moreover, this can be done in such a way that no new components are added to the intersection. After a finite number of such isotopies we can arrange for each component of $A_1 \cap A_2'$ to be an arc. Our hypothesis that C_1 and C_2 intersect minimally implies that these arcs travel from one end of A_1 to the other. Since each component of $S \setminus (C_1 \cup C_2)$ is an open disk, it follows that the boundaries of the pieces obtained by cutting open U along A_1 and A_2' are 2-spheres, and hence bound 3-balls in U. As K lies in one of these pieces, it can be isotoped into the 3-ball $P^{-1}(F_0)$.

Consider next the case where $F_0 \subset S$ is an essential non-separating annulus. Let $S_1 = \overline{S \setminus F_0}$ and choose a transverse pair of non-isotopic essential simple closed curves C_1, C_2 in S_1 which intersect minimally and such that each component of $S_1 \setminus (C_1 \cup C_2)$ is either an open disk or a noncompact annulus whose boundary is a circle component of ∂F_0. Let A_1, A_2 be the essential annuli in U associated to C_1, C_2. Without loss of generality we may suppose $K \cap A_1 = \emptyset$. Next isotope A_2, rel ∂, to an annulus $A_2' \subset U \setminus K$ which is transverse to A_1 and $P^{-1}(\partial F_0)$. It is possible, as above, to remove by isotopy all circle intersections between A_2' and the annuli $A_1, P^{-1}(\partial F_0)$. It follows that $A_2' \subset P^{-1}(\mathrm{int}(S_1))$. By construction, the closure of the components of the complement of $A_1 \cup A_2'$ in $P^{-1}(S_1)$ have boundaries which are either 2-spheres or tori which contain a component of ∂F_0. Since U is irreducible and atoroidal, the pieces of $P^{-1}(S_1)$ in this decomposition are 3-balls and solid tori on whose boundaries some component of ∂F_0 lies as a longitude. Since K lies in the complement of $A_1 \cup A_2'$, it is contained in either a 3-ball piece, and hence can be isotoped into $P^{-1}(F_0)$, or the union of $P^{-1}(F_0)$ and the two solid tori, in which case it can also be isotoped into $P^{-1}(F_0)$.

The case when $F_0 \subset S$ is an essential separating annulus is handled similarly, so we only outline the steps involved. Let S_1, S_2 be the components of $\overline{S \setminus F_0}$ and choose a transverse pair of non-isotopic essential simple closed curves C_{j1}, C_{j2} in $\mathrm{int}(S_j)$ which intersect minimally and such that each component of $S_j \setminus (C_{j1} \cup C_{j2})$ is either an open disk or a noncompact annulus whose boundary is a circle component of ∂F_0 $(j = 1, 2)$. Let A_{j1}, A_{j2} be the essential annuli in U associated to C_{j1}, C_{j2}. One first shows that K can be isotoped into the complement of $A_{11} \cup A_{21}$. Next A_{12} and A_{22} are isotoped, rel ∂, to annuli A_{12}' and A_{22}' which lie in the complement of $K \cup P^{-1}(\partial F_0)$ and which intersect $A_{11} \cup A_{21}$ in arcs running from one end of these annuli to the other. The proof is completed exactly as is done in the last case. q.e.d.

Corollary 4.3. *Let W be a compact, connected, irreducible, orientable 3-manifold and r a slope on a toral boundary component T of W. If $W(r) \cong S \times I$ is a product I-bundle, then $W(r)$ contains a new annulus.*

Proof. Since the isotopy class of an essential annulus in an I-bundle is determined by its boundary, to say that $W(r)$ contains no new annulus is equivalent to stating that any essential annulus in $W(r)$ can be isotoped into W. Thus the previous lemma implies that T is contained in a 3-ball in $W(r)$. But this is impossible as it implies W would be reducible. Thus the conclusion of the corollary must hold. q.e.d.

Our next corollary provides a recognition result for hollow products.

Corollary 4.4. *Suppose that W is a compact, connected, orientable, irreducible, atoroidal 3-manifold whose boundary contains a torus T. Let r be a slope on T. Suppose $W(r) \cong S \times I$ and γ is* *an essential simple closed curve in* $S \times \{0\}$ *or* $S \times \{1\}$*. Then either* W *is a hollow product or* $W(r)$ *contains a new annulus* A *with* $\partial A \cap \gamma = \emptyset$*.*

Proof. Identify $W(r)$ with $S \times I$ and let $P : W(r) \to S$ be the projection. Suppose that $\gamma = C \times \{0\}$ where C is an essential curve in S and that $W(r)$ contains no new annulus with boundary disjoint from γ. Since annuli in $S \times I$ with the same boundaries are isotopic, rel ∂, any essential annulus in $P^{-1}(S \setminus C)$ can be isotoped into W. Therefore Lemma 4.2 shows that we may assume T is contained in the solid torus $V = P^{-1}(N(C))$. Note that ∂V cannot compress in W, as otherwise W would be reducible. Obviously γ is a longitude of V. Since W is atoroidal, ∂V is parallel in W to T. It follows from this that W is a hollow product. q.e.d.

The interesting nature of new annuli is underscored by our next proposition.

Proposition 4.5. *Let* r, s *be slopes on a toral boundary component of a compact, connected, irreducible, orientable* 3*-manifold* W*. If* $W(r)$ *contains a new annulus* A *and* $W(s)$ *contains an essential disk* D *such that* $\partial A \cap \partial D = \emptyset$*, then* $\Delta(r, s) \leq 1$*.*

Proof. Assume $\Delta = \Delta(r, s) > 1$. Let K_r, K_s be the cores of the filled solid tori in $W(r), W(s)$ respectively. Choose A so that $|A \cap K_r|$ is minimal amongst all annuli in $W(r)$ with the same boundary, and similarly for D. Then, as usual, we get intersection graphs Γ_A, Γ_D on A, D respectively. Note that since $\partial A \cap \partial D = \emptyset$, these graphs contain no boundary edges. Suppose that Γ_A represents all types (see [17]). Then by [19, Lemma 4.4], there is a collection $\mathcal{D}$ of disk faces of Γ_A such that, if we tube D along the annuli in ∂V_s corresponding to the corners that appear in the elements of $\mathcal{D}$, and compress the resulting surface along the disks $\mathcal{D}$, then we get a disk D'. Since $\partial D' = \partial D$ and $|D' \cap K_s| < |D \cap K_s|$, this contradicts our minimality assumption on D. Hence Γ_A does not represent all types. The argument of [18, Theorem 2.5] then shows that Γ_D contains a great p-web Λ, where $p = |A \cap K_r|$. Since each of the p labels appears $\Delta \geq 2$ times at each vertex of Λ, and (by definition of a p-web) there are at most p edge endpoints at vertices of Λ that do not belong to edges of Λ, there is a label x such that Λ contains a great x-cycle. Hence Γ_D contains a Scharlemann cycle ([6, Lemma 2.6.2]). But this allows us to construct an annulus A' in $W(r)$ with $\partial A' = \partial A$ and $|A' \cap K_r| < p$, contradicting our choice of A. q.e.d.

Proposition 4.6. *Let* r, s *be slopes on a toral boundary component* *of a compact, connected, irreducible, orientable, atoroidal* 3*-manifold* W*. If* $W(r)$ *is a product* I*-bundle and* $W(s)$ *is* ∂*-reducible, then* $\Delta(r, s) = 1$*.*

Proof. By the previous proposition we may suppose that there is no new annulus in $W(r)$ whose boundary is disjoint from that of a boundary-compressing disk for $W(s)$. Corollary 4.4 and Lemma 4.1(2) now show that W is a hollow product with canonical slope s. Then Lemma 4.1(1) implies that $\Delta(r, s) = 1$. q.e.d.

5. Fillings of manifolds of large first Betti number

In this section we prove Theorem 1.2. Recall that under the hypotheses of this theorem, M is a compact, connected, orientable, hyperbolic 3-manifold whose boundary is a torus and whose first Betti number is at least 2. We noted in the introduction that if $M(r)$ is not hyperbolic, then it is either reducible, toroidal, or Seifert fibred. In the latter case we may assume that $M(r)$ is irreducible and atoroidal, and so the fact that $b_1(M(r)) \geq 1$ implies that $M(r)$ is a surface bundle over S^1 with a periodic monodromy ([22, Theorem VI.34]). The base orbifold of $M(r)$ is necessarily of the form $S^2(p, q, r)$ where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. It follows that $b_1(M(r)) = 1$ and so $b_1(M) = 2$. In particular, when $b_1(M) \geq 3$ all exceptional filling slopes are either reducible or toroidal.

A torally bounded compact 3-manifold has Euler characteristic zero, hence $b_2(M) = b_1(M) - 1 \geq 1$. We can therefore choose a closed, connected, orientable, non-separating, essential surface S in M which is Thurston norm minimizing. Since M is atoroidal, the genus of S is larger than one. According to work of Gabai [10, Corollary], there is a slope r_0 on ∂M such that for any slope $r \neq r_0$, S remains Thurston norm minimizing in $M(r)$ (in particular it is essential) and $M(r)$ is irreducible. We call the slope r_0 a *degeneracy slope* for S.

The proof of the following result is contained implicitly in [38, proof of Theorem 3.3].

Proposition 5.1. *Under our assumptions, if* $M(r)$ *is non-hyperbolic, then* $\Delta(r, r_0) \leq 1$*.*

Proof. We shall assume that the reader is familiar with terminology in [10] and [38]. By [10], there is a sequence

$$(M, \partial M) = (M_0, \delta_0) \xrightarrow{S_1} (M_1, \delta_1) \xrightarrow{S_2} \cdots \xrightarrow{S_n} (M_n, \delta_n)$$

of sutured manifold decompositions (where δ_i is the suture on M_i) such that:

- $\delta_0 = \partial M = T_0$, $S_1 = S$.
- Each (M_i, δ_i) is taut and each separating component of S_{i+1} is a product disk.
- (M_n, δ_n) is a union of a product sutured manifold and a sutured manifold $T_0 \times I$ whose suture on $\partial M = T_0 = T_0 \times 0$ is the entire torus and on $T_1 = T_0 \times 1$ is a nonempty set of annuli.

Gabai [11] associates to this sequence a branched surface B in M disjoint from ∂M which fully carries an essential lamination λ. By [38, Lemma 2.1], λ is fully carried by an essential branched surface B' which is a λ-splitting of B. Let X (resp. X') be the component of $M \setminus \mathrm{int}(N(B))$ (resp. $M \setminus \mathrm{int}\,(N(B'))$ containing $T_0 = \partial M$. Note that $X \subset X'$ since B' is a splitting of B. It follows from Gabai's construction that $X = T_0 \times I$ and its vertical boundary $\partial_v X$ is a nonempty set contained in T_1. By the definition of a sutured manifold, $\overline{T_1 \setminus \partial_v X}$ consists of two nonempty parts $\partial_+ X$ and $\partial_- X$, both of which meet each component of $\partial_v X$. It follows that $\partial_v X$ consists of at least two parallel disjoint annuli in T_1. It turns out that the core curves of these annuli are parallel in X to a curve of slope r_0 on T_0 and that B' remains essential in $M(r)$ if $r \neq r_0$. It follows that there are two essential annuli A_+, resp. A_-, in X, each having one boundary component in T_0 and the other on a component of $\partial_+ X$, resp. $\partial_- X$. Since $X \subset X'$, A_+, A_- are also essential annuli in X'.

We may assume that $r \neq r_0$ and therefore $M(r)$ is either toroidal or atoroidal, Seifert fibred. The argument of [38, Theorem 3.3] shows that $\Delta(r, r_0) \leq 1$ when $M(r)$ is toroidal, so assume $M(r)$ is atoroidal, Seifert fibred. An application of [38, Theorem 1.8 (2)] shows that if $\Delta(r, r_0) > 1$, then $X'(r)$ is not an I-bundle over a surface F with $\partial_v X'(r)$ the I-bundle over ∂F. But then by [4], $M(r)$ cannot be atoroidal Seifert fibred and hence $\Delta(r, r_0) \leq 1$ as claimed. q.e.d.

Fix slopes $r_1, r_2 \in \mathcal{E}(M)$ for which $\Delta(r_1, r_2) = \Delta(\mathcal{E}(M))$. If we can show that $\Delta(r_1, r_2) \leq 5$, then by Proposition 5.1 we have $\#\Delta(\mathcal{E}(M)) \leq 7$, and therefore the proof of Part (1) of Theorem 1.2 will be complete. Similarly it suffices to prove that $\Delta(r_1, r_2) \leq 4$ when $b_1(M) \geq 3$ to deduce Part (2).

We may assume that neither r_1 nor r_2 is r_0 by the preceding proposition. In particular we may suppose that they are not reducible filling slopes. On the other hand, if both r_1 and r_2 are toroidal filling slopes, then $\Delta(r_1, r_2) \leq 5$ in the general case ([14]), while $\Delta(r_1, r_2) \leq 4$ when $b_1(M) \geq 3$ (Theorem 3.1). Since we observed above that all exceptional filling slopes are either reducible or toroidal when $b_1(M) \geq 3$, we have completed the proof of Part (2) of Theorem 1.2. We focus on Part (1) then. From the discussion immediately above, we may assume that $M(r_1)$ is atoroidal Seifert fibred and $M(r_2)$ is either toroidal or atoroidal Seifert fibred.

Let W denote M cut along S; then ∂W consists of two copies $S_\pm$ of S, and a torus ∂M. Since M is hyperbolic and S is incompressible in M, W is irreducible, ∂-irreducible, and atoroidal. It is also ∂M-anannular in the sense that it contains no essential annulus whose boundary lies in ∂M. We use $W(r)$ to denote the manifold obtained by Dehn filling W along ∂M with slope r.

Lemma 5.2. *If W is a hollow product, then $\Delta(r_1, r_2) \leq 5$.*

Proof. Let W be a hollow product defined by an essential simple closed curve $\gamma \subset S$. Note that the canonical slope c defined in §4 is the degeneracy slope r_0 for S in our current situation. It follows from Lemma 2.1, and the discussion preceding it, that if r is a slope on ∂M such that $\Delta(r, c) = 1$, then $M(r)$ is an S-bundle over the circle. Since M is hyperbolic, there is such a slope with $M(r)$ being hyperbolic. In particular the monodromy $f : S \to S$ of the bundle $M(r) \to S^1$ is pseudo-Anosov and therefore (f, γ) fills S. Since $\Delta(r_j, c) = \Delta(r_j, r_0) = 1$ ($j = 1, 2$), there is an integer n_j such that $M(r_j) = W(T_\gamma^{n_j} f)$ (Lemma 2.1). By Theorem 2.6 we have $5 \geq |n_1 - n_2| = \Delta(r_1, r_2)$, and so we are done. q.e.d.

Recall that the genus of S is larger than 1. By hypothesis we may isotope S to be horizontal in the Seifert manifold $M(r_1)$. As S is nonseparating, it is a fibre in a realization of $M(r_1)$ as an S-bundle over S^1 ([22, Theorem VI.34]). In particular $W(r_1) \cong S \times I$ and so $W(r_1)$ contains a new annulus (Corollary 4.3). Let A be such an annulus chosen, amongst all new annuli, to have minimal intersection with ∂M. Then $A \cap W$ is an essential punctured annulus in W with boundary slope r_1.

Lemma 5.3. *If $W(r_2)$ contains an essential torus or W contains an essential punctured annulus with boundary slope r_2, then $\Delta(r_1, r_2) \leq 5$.*

Proof. When $W(r_2)$ contains an essential torus, then $\Delta(r_1, r_2) \leq 5$

by [14, Proposition 12.2]. When W contains an essential punctured annulus with boundary slope r_2, then by [14, Proposition 12.3], either $\Delta(r_1, r_2) \leq 5$ or W is a hollow product. Lemma 5.2 implies that the lemma holds in the latter case. q.e.d.

The proof of Theorem 1.2 (1) now splits into two cases. Recall that we have assumed that $r_2 \neq r_0$ and therefore S remains essential in $M(r_2)$. If $M(r_2)$ is atoroidal Seifert, then $W(r_2) \cong S \times I$ and thus contains a new annulus (Corollary 4.3). It follows that W contains an essential punctured annulus with boundary slope r_2 and thus $\Delta(r_1, r_2) \leq 5$ (Lemma 5.3). On the other hand suppose that $M(r_2)$ is toroidal. We claim that either $W(r_2)$ is toroidal or r_2 is a boundary slope associated to an essential punctured annulus lying in W. To see this, let T be an incompressible torus in $M(r_2)$ such that the lexicographically ordered pair $(|T \cap S|, |T \cap \partial M|)$ is minimal. Then $T \cap W(r_2)$ is either an incompressible torus or a disjoint union $A = \coprod_{i=1}^{n} A_i$ of essential annuli. (This follows from the minimality of $|T \cap S|$ and the incompressibility of S in $M(r_2)$.) In the latter case, let $F_i = A_i \cap W$, $1 \leq i \leq n$, and $F = A \cap W = \coprod_{i=1}^{n} F_i$. Using the minimality of $|T \cap \partial M|$, standard disk replacement arguments show that F, and hence each F_i, is incompressible and boundary incompressible in W. Since M is atoroidal, some A_i, say A_1, must meet ∂M. Then F_1 is an essential punctured annulus in W with boundary slope r_2. This proves the claim. So we may now appeal to Lemma 5.3 to get $\Delta(r_1, r_2) \leq 5$. This completes the proof of Part (1) of Theorem 1.2.

6. Singular slopes and exceptional fillings

Throughout this section we take M to be a compact, connected, orientable, torally bounded hyperbolic 3-manifold which is *large*, that is, there is a closed, essential surface $S \subset M$. Let W be the component of the exterior of S in M which contains ∂M. Evidently W is irreducible, ∂-irreducible, atoroidal and ∂M-anannular. The following fundamental theorem is due to Y.-Q. Wu.

Theorem 6.1 ([35]). *If r_1 and r_2 are two slopes on $\partial M \subset \partial W$ for which $W(r_1)$ and $W(r_2)$ are ∂-reducible, then either*

(i) $\Delta(r_1, r_2) \leq 1$, *or*

(ii) $\Delta(r_1, r_2) > 1$ *and there are a component $S \neq \partial M$ of ∂W and an annulus A properly embedded in W such that ∂A consists of an* *essential curve on S and an essential curve $C_0 \subset \partial M$. Moreover if r_0 denotes the slope of C_0 and r is a slope on ∂M, then $W(r)$ is ∂-reducible if and only if $\Delta(r_0, r) \leq 1$.*

In the rest of this section we shall make the following assumption:

$(*)$ There is a slope r_0 on ∂M such that S compresses in $M(r_0)$ but is incompressible in $M(r)$ if $\Delta(r, r_0) > 1$.

In this case we call r_0 a *singular slope* for S. For instance Wu's result guarantees that a singular slope for a given closed essential surface exists as long as that surface compresses in some filling of M. Our goal is to understand the relationship between r_0 and the set $\mathcal{E}(M)$ of exceptional filling slopes of M.

There are several situations when the existence of a closed essential surface and singular slope are guaranteed by conditions on the fillings of M. We describe two of them next.

Proposition 6.2 (cf. Theorem 2.0.3 of [6]). *Suppose that $b_1(M) = 1$ and that r_0 is a boundary slope such that $M(r_0)$ is neither a connected sum of two lens spaces nor a Haken manifold, nor $S^1 \times S^2$ if r_0 is not a strict boundary slope. Then r_0 is a singular slope of some closed essential surface in M.*

Proof. This result is essentially Theorem 2.0.3 of [6], which provides, under the conditions of the proposition, a closed essential surface (of genus larger than one) in M. It is the compressibility of the closed essential surface in $M(r_0)$ and verification of r_0 being a singular slope which must be addressed. Assume that $M(r_0)$ is neither a connected sum of two lens spaces, nor a Haken manifold, nor $S^1 \times S^2$ if r_0 is not a strict boundary slope. Choose a separating, essential surface F in M with a nonempty boundary of slope r_0 and which, subject to these conditions, has a minimal number of boundary components. In case $M(r_0) \cong S^1 \times S^2$ assume that F does not consist of fibres in some fibration of M over the circle. If F is non-planar, we can use [6, Addendum 2.2.2] and the remarks that precede it to find the desired surface, while when F is planar, we use the argument in the last paragraph of [6, p. 285]. q.e.d.

The second situation arises under the assumption of a certain type of Seifert filling of M. Let $X(G)$ denote the $PSL_2(\mathbb{C})$-character variety of a finitely generated group G. When G is the fundamental group of a path-connected space Y, we shall sometimes write $X(Y)$ for $X(\pi_1(Y))$. Note that a surjective homomorphism $G \to H$ induces an injective morphism $X(H) \to X(G)$ by precomposition. A curve $X_0 \subset X(G)$ is called *nontrivial* if it contains the character of an irreducible representation.

Each $\gamma \in G$ determines an element f_γ of the coordinate ring $\mathbb{C}[X(G)]$ where if $\rho : G \to PSL_2(\mathbb{C})$ is a representation and χ_ρ the associated point in $X(G)$, then $f_\gamma(\chi_\rho) = \mathrm{tr}(\rho(\gamma))^2$ (see, e.g., [2, §3]). When $G = \pi_1(M)$, any slope r on ∂M determines an element of $\pi_1(M)$ well-defined up to conjugation and taking inverse. Hence it induces a well-defined $f_r \in \mathbb{C}[X(M)]$.

The following theorem was announced in the introduction.

Theorem 1.7. *Let M be a compact, connected, orientable, irreducible hyperbolic 3-manifold whose boundary is a torus. Suppose that $M(r_0)$ is a big Seifert fibred manifold whose base orbifold $\mathcal{B}$ is not of the form $P^2(p,q)$. If $\mathcal{B}$ is the Klein bottle or $S^2(2,2,2,2)$, assume that $b_1(M) \geq 2$. Then r_0 is a singular slope of a closed essential surface $S \subset M$.*

Proof. First assume that the base orbifold $\mathcal{B}$ of $M(r_0)$ is hyperbolic. Corollary 13.3.7 of [31] shows that the real dimension of the Teichmüller space $\mathcal{T}(\mathcal{B})$ of $\mathcal{B}$ is at least 2. Since $X(M) \supset X(M(r_0)) \supset X(\pi_1^{\mathrm{orb}}(\mathcal{B})) \supset \mathcal{T}(\mathcal{B})$, the complex dimension of $X(M(r_0))$ is at least 1. We claim that in fact, $X(M(r_0))$ contains a nontrivial algebraic component of complex dimension 2 or more. If this were not the case, $\mathcal{T}(\mathcal{B})$ would be an open set in a nontrivial curve $X_0 \subset X(M(r_0))$. When $\chi_\rho \in \mathcal{T}(\mathcal{B})$, ρ is the holonomy of a hyperbolic structure on $\mathcal{B}$ and it is well known that if $\gamma \in \pi_1^{\mathrm{orb}}(\mathcal{B})$ has infinite order, then $f_\gamma(\chi_\rho)$ is a real number which is essentially the length of the unique geodesic in this structure representing the conjugacy class of γ (see e.g., [9, Lemme 1, p. 135]). Deforming χ_ρ in $\mathcal{T}(\mathcal{B})$ shows that $f_\gamma|X_0$ is nonconstant. But then it must take on non-real values, contrary to the fact that it is real-valued on an open subset $\mathcal{T}(\mathcal{B}) \subset X_0$.

Thus $X(M)$ has a subvariety of complex dimension 2 or larger on which f_{r_0} is constant and which contains the character of an irreducible representation. Hence if $r_1 \neq r_0$ is any other slope, it is easy to construct a nontrivial curve $X_0 \subset X(M)$ on which both f_{r_0} and f_{r_1} are constant. It follows that $f_r|X_0$ is constant for each slope r ([2, §5]) and in particular for each ideal point x of X_0 and slope r on ∂M, $f_r(x) \in \mathbb{C}$. Proposition 4.10 and Claim (p. 786) of [2] now imply that r_0 is a singular slope for a closed, essential surface in M.

The only possibilities for $\mathcal{B}$ when it is not hyperbolic are the torus T, the Klein bottle K, or $S^2(2,2,2,2)$. Note that in each case, $M(r_0)$ contains no essential surface of genus different from 1. Further, we have $b_1(M) \geq 2$: when $\mathcal{B} = T$ this is because $H_1(M) \to H_1(M(r_0)) \to H_1(\pi_1^{\mathrm{orb}}(\mathcal{B})) = \mathbb{Z}^2$ is surjective, and when $\mathcal{B} = K$, or $S^2(2,2,2,2)$, this is by hypothesis. Hence there is a Thurston norm minimizing, non-separating surface S in M whose genus is at least 2. But then S compresses in $M(r_0)$. According to [10, Corollary], S compresses in at most one filling of M, and therefore r_0 is a singular slope for S. q.e.d.

Theorem 1.5 asserts that if r_0 is a singular slope of some closed essential surface in M and r a slope on ∂M, then

$$\Delta(r_0, r) \leq \begin{cases} 1 & \text{if } M(r) \text{ is either small or reducible} \\ 1 & \text{if } M(r) \text{ is Seifert and } S \text{ does not separate} \\ 2 & \text{if } M(r) \text{ is toroidal and } \mathcal{C}(S) \text{ is infinite} \\ 3 & \text{if } M(r) \text{ is toroidal and } \mathcal{C}(S) \text{ is finite.} \end{cases}$$

The proofs of these assertions are contained in the results which follow.

Proposition 6.3. *If r_0 is a singular slope for a closed essential surface S in M and r is a reducible filling slope, then $\Delta(r, r_0) \leq 1$.*

Proof. If $\Delta(r, r_0) > 1$ then S is essential in $M(r)$ and so our hypotheses imply that $W(r)$ is reducible. According to Scharlemann [29], $(W, \partial M)$ is cabled, contrary to the fact that M is hyperbolic. Thus $\Delta(r, r_0) \leq 1$. q.e.d.

Recall that a 3-manifold which contains no closed essential surfaces is called *small*.

Corollary 6.4. *If r_0 is a singular slope for a closed essential surface S in M and r is a small filling slope, then $\Delta(r, r_0) \leq 1$.*

Proposition 6.5. *If r_0 is a singular slope for a non-separating, closed essential surface S in M and r is a Seifert filling slope, then $\Delta(r, r_0) \leq 1$.*

Proof. Suppose that $\Delta(r, r_0) > 1$, so that S remains essential in $M(r)$. Since S must be horizontal in $M(r)$ and is non-separating, $W(r) \cong S \times I$. But this contradicts Proposition 4.6. Hence $\Delta(r, r_0) \leq 1$. q.e.d.

Proposition 6.6. *Suppose that r_0 is a singular slope for a closed essential surface S in M such that $\mathcal{C}(S)$ is infinite. If r is a toroidal filling slope, then $\Delta(r, r_0) \leq 2$.*

Proof. Suppose otherwise that $\Delta(r, r_0) \geq 3$. Then S is incompressible in $M(r)$, so that $W(r)$ is ∂-irreducible. According to Proposition 6.3, $W(r)$ is irreducible and Theorem 4.1 of [38] implies that it is atoroidal. Any essential torus in $M(r)$ must intersect S, as well as ∂M. Choose one, T say, such that the lexicographically ordered pair $(|T \cap S|, |T \cap \partial M|)$ is minimal. Arguing as in the last paragraph of Section 5, we have that $T \cap W(r)$ is a set of essential annuli, $A_1, \dots, A_n$, at least one of which, say A_1, must intersect ∂M, and that if we let $F_1 = A_1 \cap W$, then F_1 is an essential punctured annulus in W.

Since $\mathcal{C}(S)$ is infinite, there is an essential annulus A in W with one boundary component in ∂M of slope r_0 and the other on $\partial W \setminus \partial M$. Isotope A in W so that $A \cap F_1$ has a minimal number of components. Then no circle component of $A \cap F_1$ can bound a disk in A or F_1. Standard cut-paste arguments also show, using (1) the essentiality of A and F_1 in W, (2) the irreducibility and ∂-irreducibility of W, (3) the minimality assumption on $|A \cap F_1|$, and (4) the essentiality of A_1 in $W(r)$, that no arc component of $A \cap F_1$ is boundary parallel in A or F_1. Thus in F_1, every arc component of $A \cap F_1$ connects an inner boundary (i.e., a component of $F_1 \cap \partial M$) to an outer boundary (i.e., a component of ∂A_1). Fix an inner boundary component of F_1 and note that since $\Delta(r_0, r) \geq 3$, there are at least three arcs of $F_1 \cap A$ incident to it. In particular two of these arcs must be incident to the same outer boundary component of F_1. These two arcs, together with two arcs in ∂F_1, cobound a disk $E_1 \subset A_1$. An innermost argument then shows that there exist two arcs a_1 and a_2 in $A \cap E_1$ which are parallel and adjacent in F_1, connecting one inner boundary and one outer boundary of F_1. Let D_2 be the disk in F_1 cobounded by a_1 and a_2. Note that the interior of D_2 is disjoint from A. The arcs a_1 and a_2 also cut off a disk D_3 from A which glues together with D_2 to form a properly embedded annulus A_* in W with one boundary component in ∂M and the other on $\partial W \setminus \partial M$. One checks from the form of the construction that the inner boundary component of A_* is an essential curve on ∂M whose slope has distance 1 from both r_0 and r. But this implies that $W \cong \partial M \times [0, 1]$ (cf. the proof of Lemma 2.5.3 of [6]), which is impossible. Hence it must be that $\Delta(r, r_0) \leq 2$. q.e.d.

Note that the proof of Proposition 6.6 actually shows that if W contains an essential annulus with one boundary component on ∂M with slope r_0, then $\Delta(r_0, r) \leq 2$ for any toroidal filling slope r of M (without the assumption that $\mathcal{C}(S)$ is infinite).

Proposition 6.7. *Suppose that r_0 is a singular slope for a closed essential surface S in M such that $\mathcal{C}(S)$ is finite. If r is a toroidal filling slope of M, then $\Delta(r, r_0) \leq 3$.*

Proof. Assume otherwise that $\Delta(r_0, r) > 3$. We will show that this leads to a contradiction.

First we may assume that W contains no essential annulus that has exactly one boundary component on ∂M. For if such an annulus exists, the finiteness of $\mathcal{C}(S)$ implies that the singular slope r_0 must be the ∂M slope of that annulus. We may then apply Proposition 6.6 and the remark following its proof to obtain a contradiction.

We plan to apply the arguments of [28] where our Proposition 6.7 was proved under the extra assumption that W be anannular. In our current setting, W may contain an essential annulus whose boundary lies in $\partial W \setminus \partial M$. This is the only new difficulty that we need pay attention to.

As in the proof of Proposition 6.6, we may assume that $W(r)$ is irreducible, atoroidal, and ∂-irreducible. Choose an essential torus T in $M(r)$ so that the lexicographically ordered pair $(|T \cap S|, |T \cap \partial M|)$ is minimal. Then $T \cap W(r)$ consists of disjoint essential annuli in $W(r)$, at least one of which intersects ∂M, and each component of $T \cap W$ is an essential surface in W.

We call a properly embedded incompressible annulus A in $W(r)$ *co-annular* if ∂A bounds an annulus in $\partial W(r)$. Note that if A is co-annular in $W(r)$ and A' is the annulus in $\partial W(r)$ with $\partial A = \partial A'$, then the torus $A \cup A'$ bounds a solid torus in the irreducible, atoroidal manifold $W(r)$. In particular, A separates $W(r)$.

Lemma 6.8. *Let A_1 be a component of $W(r) \cap T$ such that $A_1 \cap \partial M$ is nonempty. If there is an annulus A_2 in $W(r)$ such that $\partial A_2 = \partial A_1$, $\operatorname{int}(A_1) \cap \operatorname{int}(A_2) = \emptyset$, and $|A_2 \cap \partial M| < |A_1 \cap \partial M|$, then $W(r) \cap T$ has a component which is co-annular in $W(r)$ and has nonempty intersection with ∂M.*

Proof. We may assume that A_2 has been selected, amongst all annuli satisfying the conditions of the lemma, to minimize the lexicographically ordered pair $(|A_2 \cap \partial M|, |A_2 \cap T|)$. Since $W(r)$ is irreducible and atoroidal, the torus $A_1 \cup A_2$ bounds a solid torus V_* in $W(r)$ such that $T \cap V_*$ is a set of disjoint essential annuli. Let A_* be one such annulus that is outermost toward A_2, i.e., if A' in A_2 is the annulus bounded by ∂A_*, then $A_* \cup A'$ bounds a solid torus $V' \subset V_*$ whose interior is disjoint from T. Observe that $|A_* \cap \partial M| > |A' \cap \partial M|$. When $\mathrm{int}(A_2) \cap T = \emptyset$ this follows from our hypotheses. On the other hand, when $\mathrm{int}(A_2) \cap T \neq \emptyset$, if $|A_* \cap \partial M| \leq |A' \cap \partial M|$ we could replace A' in A_2 by A_* to obtain, after a small isotopy, an annulus A_2' which has all the properties of A_2 listed in the statement of the lemma and which satisfies $(|A_2' \cap \partial M|, |A_2' \cap T|) < (|A_2 \cap \partial M|, |A_2 \cap T|)$, contrary to our choices.

Now let T' be the torus in $M(r)$ obtained from T by replacing the annulus $A_* \subset T$ by A'. Then $|T \cap S| = |T' \cap S|$ while $|T' \cap \partial M| < |T \cap \partial M|$. Therefore T' bounds a solid torus V'' in $M(r)$. The intersection $S \cap V''$ consists of a set of incompressible annuli in V'' since S is incompressible in $M(r)$. Every such annulus is boundary parallel in V''. Let A_3 be an outermost such annulus and let A_4 be the annulus in $\partial V''$ which is parallel to A_3 in V''. Since every component of $T \cap W(r)$, resp. $T \cap (\overline{M \setminus W})$, is essential in $W(r)$, resp. $\overline{M \setminus W}$, A_4 must contain the annulus A'. Now let A_5 be the annulus obtained from A_4 by replacing A' by A_*. Then it is evident that A_5 is a component of $W(r) \cap T$ which is co-annular and intersects ∂M. q.e.d.

Now we choose a component A of $W(r) \cap T$ in such a way that if $W(r) \cap T$ contains a co-annular component which intersects ∂M, then A is such a component, and if $W(r) \cap T$ contains no co-annular components which intersect ∂M, then A is any component of $W(r) \cap T$ which has nonempty intersection with ∂M. Suppose that $A \cap \partial M$ has n components. Then $n > 0$. Let $F_2 = A \cap W$. Then F_2 is an essential punctured annulus in W.

Amongst all compressing disks in $W(r_0)$, let D be one which intersects ∂M minimally, say with m intersection components. Let $F_1 = D \cap W$. Then F_1 is an essential punctured disk in W. We have assumed that $m > 1$. As usual, we may assume, up to isotopy in W, that $F_1 \cap F_2$ contains no circle component that bounds a disk in F_1 or F_2, and no arc component that is ∂-parallel in F_1 or F_2 (cf. the proof of Proposition 6.6). Again we have two intersection graphs: Γ_1 in the disk D and Γ_2 in the annulus A.

Lemma 6.9. *If Γ_1 has a Scharlemann cycle, then A is co-annular.*

Proof. Let D' be the Scharlemann disk bounded by the Scharlemann cycle with label pair, say $\{1, 2\}$. Let U be a regular neighborhood of $A \cup H_{12} \cup D'$ in $W(r)$. Then ∂U is a torus. The annulus A may be considered as an annulus in ∂U and $A' = \partial U \setminus \mathrm{int}(A)$ is an annulus such that $|A' \cap \partial M| = |A \cap \partial M| - 2$. Hence by Lemma 6.8 and our choice of

A, A must be co-annular. q.e.d.

Lemma 6.10. Γ_1 *cannot have two Scharlemann cycles with different label pairs.*

Proof. Suppose that Γ_1 has two Scharlemann disks with different label pairs, say $\{i, i+1\}$ and $\{j, j+1\}$. Let D_1 and D_2 be the Scharlemann disks bounded by the two Scharlemann cycles respectively. Let U_1 be a regular neighborhood of $A \cup H_{i,i+1} \cup D_1$ in $W(r)$. Then U_1 is a solid torus in the irreducible, atoroidal manifold $W(r)$. Also, considering A as lying in ∂U_1, the annulus $A' = \partial U_1 \setminus \mathrm{int}(A)$ is not parallel to A through U_1. Similarly construct U_2 from the other disk D_2.

Now if D_1 and D_2 lie on different sides of A, then it is easy to see that $U_1 \cup U_2$ is a Seifert fibered space over the disk with exactly two cone points and thus $\partial(U_1 \cup U_2)$ is incompressible in $U_1 \cup U_2$. Since $W(r)$ is assumed to be irreducible and ∂-irreducible, $\partial(U_1 \cup U_2)$ must be incompressible in $W(r)$ as well. Thus it is an essential torus in $W(r)$. But this contradicts our assumption that $W(r)$ is atoroidal. On the other hand if D_1 and D_2 are on the same side of A, then their label pairs must be disjoint. Let U be a regular neighborhood of $A \cup H_{i,i+1} \cup D_1 \cup H_{j,j+1} \cup D_2$ in $W(r)$. Again it is easy to see that U is a Seifert fibered space over the disk with exactly two cone points, which yields a contradiction as in the former case. Thus the lemma holds. q.e.d.

Lemmas 2.1-2.6 of [28] each hold in our current situation. (Lemma 2.1 (5) and Lemma 2.3 of [28] follow from our Lemma 6.9; Lemma 2.1 (3) of [28] is reproved here as Lemma 6.10; Lemma 2.6 of [28] holds because we have assumed that W contains no embedded annulus with exactly one boundary component contained in ∂M; the rest of the results of §2 of [28] hold in our setting with proofs identical to those given there). Arguing exactly as in §3 of [28], one obtains a contradiction. We have now completed the proof of Proposition 6.7. q.e.d.

7. Fillings of large manifolds of first Betti number 1

In this section we shall assume that M is large and has first Betti number 1. Our goal is to prove Theorem 1.9:

(1) If there is a closed, essential surface $S \subset M$ such that $\mathcal{C}(S)$ is finite, then $\Delta(\mathcal{E}(M)) \leq 5$.

(2) If there are at least two different slopes on ∂M each of which is a

singular slope of an essential closed surface, then $\Delta(\mathcal{E}(M)) \leq 5$.

(3) If there are at least two different slopes on ∂M each of which is either a singular slope of an essential closed surface or a degeneracy slope of an essential branched surface, then $\Delta(\mathcal{E}_{TOP}(M)) \leq 5$.

Proof of Part (1). Suppose that M is a large hyperbolic manifold with $b_1(M) = 1$. Suppose that S is a closed essential surface in M. The slopes in $\mathcal{E}(M)$ are partitioned into two groups. The first consists of those slopes $r \in \mathcal{E}(M)$ which either lie in $\mathcal{C}(S)$ or for which $M(r)$ is reducible. The second group are the slopes in $\mathcal{E}(M) \setminus \mathcal{C}(S)$ whose fillings are irreducible. In particular these fillings are Haken and therefore satisfy the geometrisation conjecture [32]. Hence they are either toroidal or Seifert. We claim that the Seifert filling slopes in this group are also toroidal. To see this, suppose that S stays essential in the Seifert filling $M(r)$. Isotope S to a horizontal surface in $M(r)$. As $b_1(M) = 1$, S separates M and so splits $M(r)$ into the union of two twisted I-bundles over non-orientable surfaces. It follows that the surface underlying the base orbifold $\mathcal{B}$ of $M(r)$ is also non-orientable. Hence $M(r)$ is toroidal unless $\mathcal{B} = \mathbb{R}P^2$ or $\mathbb{R}P^2(p)$. But the latter cannot occur as the corresponding Seifert manifolds have no closed essential surface of genus larger than one. Hence $\mathcal{E}(M)$ is contained in the union of $\mathcal{C}(S)$ with the set of reducible or toroidal filling slopes. As $\mathcal{C}(S)$ is finite, Wu's theorem (Theorem 6.1) shows that $\Delta(\mathcal{C}(S)) \leq 1$, while $\Delta(r_1, r_2) \leq 5$ if r_1, r_2 are either reducible or toroidal filling slopes by [19], [14], [26] and [37]. Finally since $\Delta(\mathcal{C}(S)) \leq 1$, each slope $r_0 \in \mathcal{C}(S)$ is a singular slope for S. An appeal to Corollary 1.6 finishes the proof.

Proof of Part (2). If there is a closed, essential surface S in M for which $\mathcal{C}(S)$ is finite, then the desired conclusion follows from what we have just proved. Assume then that $\mathcal{C}(S)$ is infinite for each closed, essential surface S in M. In particular, each such surface uniquely determines a singular slope r_0 on ∂M. According to Corollary 1.6 we have $\Delta(r_0, r) \leq 2$ for each $r \in \mathcal{E}(M)$. The proof is completed by applying the following easily verified fact: if r_1, r_2 are distinct slopes on ∂M and $\mathcal{S}$ is the set of slopes of distance no more than 2 from r_1 and r_2, then $\Delta(\mathcal{S}) \leq 5$.

Proof of Part (3). The proof is similar to that of Part (2). In this case we also need to apply Theorem 1.8. q.e.d.

8. Seifert surgery on hyperbolic knots in the 3-sphere

Suppose that K is a hyperbolic knot in the 3-sphere with exterior M_K. Suppose further that r is a non-meridional slope on ∂M_K such that $M_K(r)$ is Seifert fibred. Theorem 1.11 states that:

(1) If K is a small knot, then $M_K(r)$ is not a very big Seifert manifold.

(2) If r_0 is a singular slope of an essential closed surface in M_K, then $\Delta(r_0, \mu_K) \le 1$ and $\Delta(r_0, r) \le 1$.

(3) If μ_K is a singular slope of an essential closed surface in M_K, then r is an integral slope. In particular, this occurs if either μ_K is a boundary slope or there is an essential closed surface S in M_K such that $\mathcal{C}(S)$ is finite.

(4) If K admits a very big Seifert surgery slope r_0, then r_0 is integral and $\Delta(r_0, r) \le 1$. Hence K admits no more than two very big Seifert surgeries, and if two, then:

- They correspond to successive integral slopes.
- If r is non-integral, it is half-integral.

(5) If K is amphicheiral and $M_K(r)$ is a big Seifert manifold, then K is fibred and r is the longitudinal slope. We prove these assertions one by one.

Proof of Part (1). When K is small, the dimension of $X(M_K)$, the $PSL_2(\mathbb{C})$-character variety of $\pi_1(M_K)$, is at most one ([5, Proposition 2.4]). If $\mathcal{B}$ denotes the base orbifold of $M_K(r)$, then $X(\pi_1^{\mathrm{orb}}(\mathcal{B})) \subset X(M_K(r)) \subset X(M_K)$. Hence the dimension of $X(\pi_1^{\mathrm{orb}}(\mathcal{B}))$ is bounded above by 1. On the other hand, if $\mathcal{B}$ is hyperbolic with Teichmüller space $\mathcal{T}(\mathcal{B})$, there is a sequence of inclusions

$$\mathcal{T}(\mathcal{B}) \subset X(\pi_1^{\mathrm{orb}}(\mathcal{B})) \subset X(M_K(r)) \subset X(M_K).$$

Thus $\dim_{\mathbb{R}} \mathcal{T}(\mathcal{B}) \le 2$. Since $H_1(M_K(r))$ is cyclic and $M_K(r)$ is very big, the base orbifold $\mathcal{B}$ is of the form $S^2(p_1, \ldots, p_n)$ with $n \ge 4$ and $\max\{p_1, p_2, p_3, p_4\} > 2$, or $P^2(p_1, \ldots, p_n)$ with $n \ge 3$. In the latter case one can verify that $\mathcal{B}$ is hyperbolic (cf. [31, Theorem 13.3.6]). Further $\dim_{\mathbb{R}} \mathcal{T}(\mathcal{B}) = 2n - 3 \ge 3$ ([31, Corollary 13.3.7]), which is impossible. In the former case, $\mathcal{B}$ is again hyperbolic. The real dimension $\mathcal{T}(S^2(p_1, \ldots, p_n))$ is given by $2n - 6 \ge 2$ and so $\mathcal{B}$ has the form

$S^2(p_1, p_2, p_3, p_4)$. Furthermore, since the complex dimension of $X(M_K)$ is 1, $\mathcal{T}(\mathcal{B})$ contains an open subset of an algebraic component X_0 of $X(\pi_1^{\mathrm{orb}}(\mathcal{B})) \subset X(M_K)$. But the argument in the proof of Theorem 1.7 shows that this is false. Hence $M_K(r)$ is not a very big Seifert manifold.

Proof of Part (2). If r_0 is a singular slope of an essential closed surface $S \subset M_K$, then $\Delta(r_0, \mu_K) \leq 1$ as $\mu_K \in \mathcal{C}(S)$. In fact $r \in \mathcal{C}(S)$ as well. To see this, suppose otherwise. Then S is isotopic to a horizontal incompressible surface in $M_K(r)$ and therefore is either non-separating or splits $M_K(r)$ into two twisted I-bundles over a closed non-orientable surface. Neither possibility can arise in our situation as a closed surface in S^3 is necessarily orientable and separating. Now that $r \in \mathcal{C}(S)$, we automatically have $\Delta(r_0, r) \leq 1$.

Proof of Part (3). The first assertion follows from Part (2). Proposition 6.2 implies that μ_K is a singular slope of a closed essential surface in M_K if μ_K is a boundary slope. Also, if S is an essential closed surface in M_K and $\mathcal{C}(S)$ is finite, each of its slopes is a singular slope of S. In particular this is true for μ_K.

Proof of Part (4). Let r_i be a very big Seifert surgery slope on ∂M_K. By Theorem 1.7, r_i is a singular slope of some closed essential surface S_i in M_K and therefore an appeal to Part (2) completes the proof.

Proof of Part (5). If r is a very big Seifert surgery slope of K, then so is its image slope under an orientation reversing diffeomorphism of M_K. By Part (3), the only possibility is for r to be the longitudinal slope of K. Then $M_K(r)$ admits a closed non-separating essential surface. It follows that $M_K(r)$ fibres over the circle ([22, Theorem VI.34]) and so K is a fibred knot [11, Corollary 8.19].

If r is a big Seifert surgery slope, but not a very big one, then the base orbifold of $M_K(r)$ is either $K, S^2(2,2,2,2)$, or $P^2(p,q)$. The first two are ruled out by homological considerations, while Motegi has proved that the last one is impossible ([25, Theorem 1.3]). q.e.d.

9. Examples

We begin by constructing infinitely many examples which show that the first inequality in Theorem 1.5 is sharp.

Example 9.1 (a) Let K be an arborescent knot K of type II with exterior M_K and meridional slope μ_K. Wu shows that if $\Delta(r, \mu_K) = 1$ and r is not a boundary slope, then $M_K(r)$ is small [36]. Then μ_K is the singular slope of a closed essential surface in M_K (Theorem 6.1), and thus the distance between a singular slope and a small filling slope can be 1.

(b) Eudave-Munoz and Wu [7] generalized work of Gordon and Litherland [16] to produce infinitely many hyperbolic 3-manifolds W_p ($p \geq 2$) which admit two distinct reducible fillings. Fix $p \geq 2$ and set $W = W_p$. The boundary of W is a union of two tori T_1, T_2 and there are distinct slopes r_1, r_2 on T_1 such that $W(r_1) \cong P^3 \# Q_1$ and $W(r_2) \cong P^3 \# Q_2$ where Q_1, Q_2 admit Seifert structures whose base orbifolds are of the form $D^2(2,2), D^2(2p,2p)$ respectively. If s_j is the slope on $\partial Q_j = T_2$ represented by a fibre of the Seifert structure, then $Q(r)$ has a nonabelian fundamental group whenever $\Delta(r, s_j) \geq 2$. Choose a slope r on T_2 such that $M = W(r)$ is hyperbolic and $\Delta(r, s_1), \Delta(r, s_2) \geq 2$. It is not hard to see that M has first Betti number 1, and therefore we can apply Theorem 2.0.3 of [6] to the boundary slopes r_1, r_2 to see that both are singular slopes of closed essential surfaces in M. Hence the distance between a singular slope and a reducible filling slope can be 1.

Our next example shows that the second inequality in Theorem 1.5 is sharp.

Example 9.2 Let P be a closed orientable irreducible atoroidal Seifert fibred manifold with a non-separating closed incompressible surface S of genus larger than one. Then P is a S-bundle over S^1. Let $f : S \to S$ be the monodromy of the bundle, which is an irreducible periodic diffeomorphism. Note that for any simple closed essential curve K in S, (f, K) fills S. Let $M = P - \operatorname{int}(N(K))$. Then $b_1(M) = 2$. There is a closed essential surface S_* in M, which is a parallel copy of S in P. The surface S_* cuts M into a hollow product W. Let c be the canonical slope of W and let μ be the meridian slope of K. Then by Theorem 2.3 and (the proof of) Lemma 2.1, $M(r)$ are hyperbolic S_*-bundles over S^1 for most of the slopes r with $\Delta(c, r) = 1$. Thus for such a slope r, $M(r)$ has a pseudo-Anosov monodromy. Also the core curve K_r of the filling solid torus is isotopic to an essential closed curve in a surface fibre of $M(r)$. So it follows from Lemma 2.2 that M is hyperbolic. Note that c is the singular slope of S_* and obviously $\mu \neq c$ is a Seifert filling slope.

Our next example provides a family of infinitely many hyperbolic M_n with $b_1(M_n) = 2$, $\#(\mathcal{E}(M_n)) \geq 4$ and $\Delta(\mathcal{E}(M_n)) \geq 2$ (cf. Theorem 1.2).

Example 9.3 Consider the $(2, 2, n)$-pretzel link in S^3 (Figure 1),

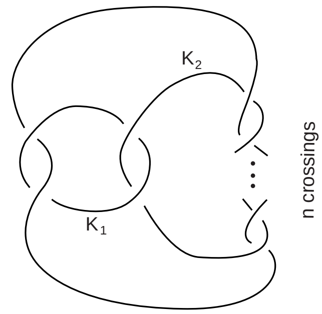


Figure 1: $(2,2,n)$-pretzel link.

where $n > 1$ is an odd integer. The link consists of two components, one, denoted K_1, being the trivial knot and the other, denoted K_2, the $(2,n)$-torus knot. It follows that the link is not a torus link (since its two components are not isotopic to each other in S^3). Let Y_n be the exterior of the link and let T_i be its torus boundary component corresponding to K_i. On each T_i slopes are parameterized by standard meridian-longitude coordinates.

We first show that Y_n is hyperbolic. Note that the link is alternating. Thus by [24] we only need to show that the link is non-split and prime. With a single application of Kirby-Rolfsen surgery calculus, we see that $Y_n(T_1, 1/k)$ is a hyperbolic 2-bridge knot exterior for all k large (cf. [21]). It follows directly that the link is non-split (for otherwise $Y_n(T_1, 1/k)$ should always be the $(2,n)$-torus knot exterior). It also follows that the link is prime. For otherwise, Y_n contains an essential torus T which bounds a solid torus V in S^3 such that there is a meridian disk of V which intersects the link in exactly one point. This torus must be compressible in $Y_n(T_1, 1/k)$ for all k. So T and T_1 bound a cable space, and thus any meridian disk of the solid torus V must intersect K_1 at least twice, giving a contradiction.

Next we show that $M_n = Y_n(T_2, 0)$ is hyperbolic. Note further that for large k, $Y_n(T_1, 1/k)$ is the exterior of a hyperbolic 2-bridge knot exterior whose 0-slope is not the boundary slope of essential punctured sphere or torus [21]. Thus $Y_n(T_1, 1/k; T_2, 0)$ is a hyperbolic manifold for all large k. It follows that if $Y_n(T_2, 0)$ is reducible, then it must be a connected sum of a closed hyperbolic manifold and a solid torus whose meridian slope is the 0-slope on T_1. But then $Y_n(T_1, 1/0; T_2, 0)$ is also hyperbolic, contradicting the fact that $Y_n(T_1, 1/0; T_2, 0)$ is the same manifold obtained by Dehn surgery on S^3 along the $(2,n)$-torus knot with the 0-slope, which is Seifert fibred. It also follows that if $Y_n(T_2,0)$ contains an essential torus, then it is cabled and the slope of the cabling annulus is the 0-slope on T_1. In other words, if $Y_n(T_2,0)$ contains an essential torus, then $Y_n(T_1,0;T_2,0)$ contains a lens space summand. But this is impossible since $\mathrm{lk}(K_1,K_2)=0$, and therefore the first homology of $Y_n(T_1,0;T_2,0)$ with $\mathbb{Z}$-coefficients is $\mathbb{Z}\oplus\mathbb{Z}$. Noting that $b_1(Y_n(T_2,0))=2$, $Y_n(T_2,0)$ must be hyperbolic.

Finally we show that $b_1(M_n)=2$, and $\mathcal{E}(M_n)\supset\{0,1,2,1/0\}$. Again since the linking number of K_1 and K_2 is zero, $H_1(M_n;\mathbb{Z})=\mathbb{Z}\oplus\mathbb{Z}$. We have just noted that $M_n(1/0)$ is the same manifold as that obtained by 0-Dehn surgery on S^3 along the $(2,n)$-torus knot, and thus is a Seifert fibred manifold. From the standard link diagram of $K_1\cup K_2$, we see that there is an once punctured torus in Y_n with its boundary in T_1 with slope 0. It follows that $M_n(0)$ contains a non-separating torus and thus $M_n(0)$ is not hyperbolic. Again by Kirby-Rolfsen surgery calculus, $M_n(1)$ is the same manifold as that obtained by Dehn surgery on S^3 along the $(2,n+2)$-torus knot with the 0-slope. Thus $M_n(1)$ is Seifert fibred. Also from the link diagram of $K_1\cup K_2$, we see that there is a twice punctured Klein bottle with one boundary component on T_1 with slope 2 and the other boundary component on T_2 with slope 0 (a spanning surface of the link). Thus $M_n(2)$ contains a Klein bottle and so is not hyperbolic.

Note that the meridian slope of M_n must be the degeneracy slope for M_n defined in Section 5 since the unique non-separating essential closed surface in $M_n(1)$ has genus larger than that of the unique non-separating essential closed surface in $M_n(1/0)$.

We suspect that $\mathcal{E}(M_n)$ is precisely the set $\{0,1,2,1/0\}$. For a fixed n, this can be checked using the SnapPea program. By Theorem 1.2 and Proposition 5.1, we only need to check the slopes $3,4,5,-1,-2,-3$.

The following example shows that, for any $n\geq 2$, there is a hyperbolic M such that $b_1(M)\geq n$, having two toroidal filling slopes r_1 and r_2 with $\Delta(r_1,r_2)=2$ (cf. Theorem 1.2 and Theorem 3.1).

Example 9.4 Let P be a pair of pants, and let K be the knot in $P\times I$ shown in Figure 2. Let $X=P\times I-\mathrm{int}\,(N(K))$ be the exterior of K. Then ∂X has two components, a torus $T_0=\partial N(K)$, and a genus two surface $P_0\cup P_1\cup(\cup_{i=1}^3 A_i)$, where $P_i=P\times\{i\}$, $i=0,1$, and A_1,A_2,A_3 are annuli. Let C_i denote a core of A_i, $i=1,2,3$. It is easy

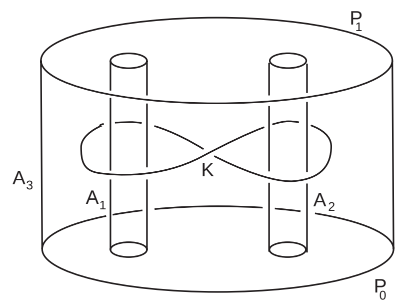


Figure 2: The knot K in $P \times I$.

to show that:

(1.1) X is irreducible.

(1.2) X is atoroidal.

(1.3) Any incompressible annulus A in X with $\partial A \subset P_1 \cup P_2$ is parallel into ∂X.

Also, parameterizing slopes on T_0 in the obvious way, we have (see [15, proof of Theorem 5.3]):

(1.4) $X(0)$ contains an annulus A_0 with $\partial A_0 = C_1 \cup C_2$, and $X(2)$ contains a Möbius band B with $\partial B = C_3$.

Let W be a compact, orientable, irreducible, ∂-irreducible, orientable, anannular 3-manifold with ∂W a surface of genus 2, and $b_1(W) \geq n+1$. Decompose ∂W as $P \cup_\partial Q$, where P and Q are pairs of pants. Let W_0, W_1 be copies of W, and let $Y = X \cup W_0 \cup W_1$, where W_i is glued to X along P_i, $i = 0, 1$. Then $\partial Y = T_0 \cup S$, where S is the genus 2 surface $Q_0 \cup Q_1 \cup (\cup_{i=1}^3 A_i)$. Note that P_i is incompressible in Y, $i = 0, 1$. It also follows easily, using (1.1) and (1.2) above, and the properties of W, that:

(2.1) S is incompressible in Y.

(2.2) Y is irreducible.

(2.3) Y is atoroidal.

Note that we still have $A_0 \subset Y(0)$ and $B \subset Y(2)$.

(2.4) There is no essential annulus A in Y with ∂A contained in S.

Proof. Let A be such an annulus. We may assume that $A \cap (P_0 \cup P_1)$ is a disjoint union of circles and properly embedded arcs, and that no circle component bounds a disk in either A or $P_0 \cup P_1$.

If $A \cap (P_0 \cup P_1)$ has an arc component that is boundary parallel in A, let α be an outermost such, cutting off a disk $D \subset A$. Then D is contained in either X or (say) W_0. But in both cases it is clear that we may isotope A to eliminate α. Hence we may assume that $A \cap (P_0 \cup P_1)$ consists of either circles that are cores of A, or arcs with one endpoint on each component of ∂A.

In the first case, using (1.3) above and the fact that W is anannular, we see that A is parallel into ∂Y, a contradiction. (This includes the case when $A \cap (P_0 \cup P_1) = \emptyset$.)

In the second case, there is an adjacent pair of arcs α_1, α_2 on A which cut off a disk D that lies in (say) W_0. Thus $\partial D = \alpha_1 \cup \beta_1 \cup \alpha_2 \cup \beta_2$, where β_1 and β_2 are arcs in $\partial A \cap Q_0$. Since W_0 is ∂-irreducible, either the arcs β_1 and β_2 are boundary parallel in Q_0, or the arcs α_1 and α_2 are boundary parallel in P_0. In both cases, we may isotope A to reduce $|\partial A \cap (P_0 \cup P_1)|$, (in the second case using the boundary incompressibility of A). q.e.d.

Let $Z = S \times I \cup H \cup V$ be defined as follows. Here H is a *round 1-handle*, $H \cong S^1 \times I \times I$, attached along $S^1 \times I \times \{0, 1\}$ to $A_1 \cup A_2$ in $S \times \{1\}$, and V is a solid torus, attached along a $(2, 1)$-annulus in ∂V to A_3 in $S \times \{1\}$. Then $\partial Z = S \cup S'$, where $S = S \times \{0\}$ and $S' \cong S$. It is not hard to show:

(3.1) Z is ∂-irreducible.

(3.2) Z is irreducible.

(3.3) Z is atoroidal.

Let W' be another copy of W, and let $U = Z \cup_{S'} W'$. Then, from (3.1), (3.2), (3.3) and the properties of W, we have:

(4.1) U is ∂-irreducible.

(4.2) U is irreducible.

(4.3) U is atoroidal.

Finally, define $M = Y \cup_S U$. Then:

(5.1) M is irreducible.

(5.2) M is atoroidal.

(5.3) $b_1(M) \geq n \geq 2$.

Thus M is hyperbolic. Let A_0' be an extension of the core annulus of the round 1-handle $H \subset Z$, with $\partial H = C_1 \cup C_2$. Then $M(0)$ contains $A_0 \cup_\partial A_0'$. We can choose the attaching map of H so that $A_0 \cup_\partial A_0'$ is a Klein bottle.

Note also that $C_3 \subset S \times \{0\}$ bounds a Möbius band B' in Z. Hence $M(2)$ contains the Klein bottle $B \cup_\partial B'$. Hence each of $M(0)$, $M(2)$ is either toroidal or reducible. But any two reducible fillings have distance at most 1 [19], and a reducible filling and a toroidal filling on a large hyperbolic manifold have distance at most 1 [38]. Hence $M(0)$ and $M(2)$ are both toroidal.

Appendix

Let W be the exterior of the Whitehead link pictured in Figure 3 and r a slope on a boundary component of W. We will denote by M_r the r Dehn filling of W. Since there is an isotopy of S^3 which interchanges the two boundary components of W, we have

$$M_r(s) \cong M_s(r).$$

Identify the slopes on either component of ∂W with $\mathbb{Q} \cup \{\frac{1}{0}\}$ in the usual way.

Proposition. *For each slope $r \neq 0, 4$ on a boundary component of W, the manifold M_r contains no closed, essential surface.*

Proof. Assume that M_r contains a closed, essential surface S. From above we have $M_r(\frac{1}{n}) = M_{\frac{1}{n}}(r)$ for each $n \in \mathbb{Z}$. It can easily be seen that $M_{\frac{1}{n}}$ is the exterior of the 2-bridge knot corresponding to the rational fraction $\frac{-2}{4n-1}$. In particular it is small [21] and therefore so is $M_{\frac{1}{n}}(r)$ as long as r is not a boundary slope of $M_{\frac{1}{n}}$. Again by [21] we see that $r \neq 0, 4$ is a boundary slope of $M_{\frac{1}{n}}$ for at most one n. Hence S compresses in $M_r(\frac{1}{n})$ for infinitely many n. It follows by Wu's theorem (Theorem 6.1) that S is incompressible in $M_r(\frac{m}{n})$ as long as $|m| > 1$.

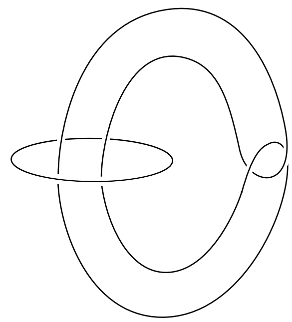


Figure 3: The Whitehead link.

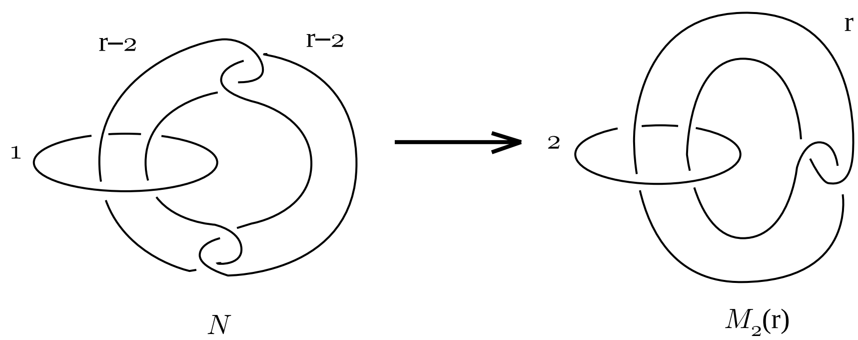


Figure 4: A double cover.

This is true, in particular, for $M_r(2) = M_2(r)$. We show that this is not the case.

The double cover of the exterior of the "horizontal" component of the Whitehead link restricts to a double cover of W, and subsequently induces the double cover $N \to M_2(r)$ depicted in Figure 4. Blowing down the component labeled "1" shows that N is homeomorphic to the manifold obtained by performing $r-2$ surgery on both components of $L_{2,4}$, the $(2,4)$ torus link (Figure 5). The exterior of $L_{2,4}$ is a Seifert manifold whose base orbifold is an annulus with exactly one cone point, and its order is 2. Moreover, since $r \neq 4$, the distance d between the slopes $r-2$ and 2, the fibre of the Seifert structure on the exterior of $L_{2,4}$, is nonzero. Thus N is a Seifert manifold with base orbifold $S^2(2,d,d)$. The first homology of $N \cong L_{2,4}(r-2,r-2)$ has order $|r(r-4)|$, and therefore our constraints on r imply that N is a small manifold ([22, VI.13]). But this contradicts the fact that the inverse image of S in N is an essential closed surface. Thus the manifold M_r is small. q.e.d.

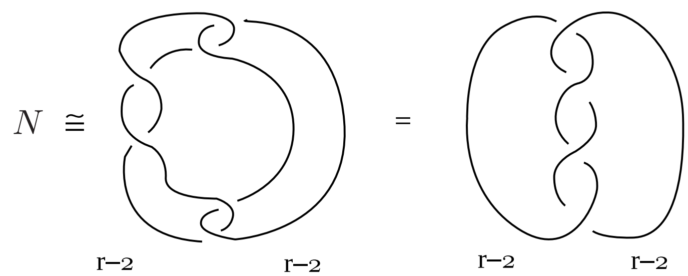


Figure 5: N as surgery on a torus link.

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